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Third-Order, Minimal-Parameter Solution of the Orthogonal Matrix Differential Equation

Margalit Ronen* and Yaakov Oshman†
Technion—Israel Institute of Technology, Haifa 32000, Israel

The problem of minimal-parameter solution of the orthogonal matrix differential equation is addressed. This well-known equation arises naturally in three-dimensional attitude determination problems (in aircraft and satellite navigation systems), as well as in the square-root solution of the matrix Riccati differential equation. A direct solution of this equation involves n^2 integrations for the elements of the nth-order solution matrix. However, since an orthogonal matrix is determined by only $n(n-1)/2$ independent (albeit nonunique) parameters, a much more efficient solution may, conceivably, be obtained by a parametrization of the problem in terms of such parameters. A new, third-order minimal parametrization, which is motivated by the Peano–Baker solution of linear matrix differential equations, is introduced. The parameters and their corresponding differential equation are very simple and natural. The proposed method is used to provide a new derivation of a closed-form third-order quaternion propagation algorithm, which is commonly used in strapdown inertial navigation systems utilizing rate-integrating gyros. A numerical example is used to demonstrate the viability and high efficiency of the new algorithm.

Introduction

THIS paper is concerned with the minimal-parameter solution of the well-known orthogonal matrix differential equation

$$\dot{V}(t) = W(t)V(t), \qquad V(t_0) = V_0 \tag{1}$$

where $V(t) \in \mathcal{R}^{n,n}$ is orthogonal, $W(t)$ is skew-symmetric for all $t \geq t_0$, and the raised dot indicates the temporal derivative. This equation arises naturally in three-dimensional attitude determination problems, as well as in the square-root solution of the matrix differential Riccati equation.[1] The properties of V and W enable a minimal-parameter solution, which should, conceivably, be much more efficient than a direct solution, based on n^2 straightforward integrations as implied by Eq. (1).

To the best of the authors' knowledge, the problem of minimal-parameter solution of Eq. (1) was originally suggested by Oshman and Bar-Itzhack.[1] The foundation of the problem lies in the observation that, although the number of scalar integrations implied by Eq. (1) is n^2, the orthogonality of V may be used to introduce $n(n+1)/2$ relations among its elements. Hence, the n^2 elements of V are functions of only $m = n(n-1)/2$ independent parameters. A considerable reduction of the computational burden, involved in the solution of Eq. (1), can thus be achieved by parametrizing the matrix V in terms of m such independent parameters, solving a differential equation for these parameters only, and then algebraically transforming the parameters into V.

In their recent investigation of the minimal-parameter problem, Bar-Itzhack and Markley[2,3] proved that if the matrix W appearing in Eq. (1) is skew-symmetric and the initial condition matrix is orthogonal, then the solution $V(t)$ is also orthogonal for all $t \geq t_0$. They also proved that any time-varying orthogonal matrix $V(t)$ satisfies a matrix differential equation having the form of Eq. (1), for some skew-symmetric matrix W. To find an appropriate parametrization, Bar-Itzhack and Markley used the observation that, for $n = 3$, Eq. (1) is identical to the well-known differential equation of the transformation matrix in three-dimensional Euclidean space. That matrix is, of course, also orthogonal, W being a skew-symmetric matrix whose entries are the three components of the angular velocity vector at which the body rotates with respect to some reference coordinate system. Hence, the original n-dimensional minimal-parameter problem may be considered an extension of the three-dimensional attitude determination problem and, conversely, the latter is a special case of the problem at hand. Using these observations, Bar-Itzhack and Markley explored the possibility of finding a parametrization method for the n-dimensional case of Eq. (1), based on extension of various known parametrizations of the three-dimensional transformation matrix. Three such methods were investigated in Ref. 3, based on Euler angles, quaternions, and Rodrigues parameters (also known as the Gibbs vector[4]). Only the last method was found effective for extension, and a minimal-parameter solution of Eq. (1), based on extended Rodrigues parameters (ERPs), was presented and demonstrated using a numerical example.

The approach taken in Ref. 3 utilizes physical insight and reasoning to mathematically extend an existing three-dimensional method—namely the Rodrigues parameters—into $\mathcal{R}^n$. A different approach is taken in this paper, which is motivated by the Peano–Baker method for the solution of linear matrix differential equations.[5] This results in a new minimal set of parameters, which are used to provide a third-order solution to Eq. (1). It is shown that these parameters, and their corresponding differential equation, are very simple and natural to the problem. A numerical example is used to demonstrate the viability of the method and to compare the accuracy and efficiency of the solution based on these parameters with those of the exact solution and the ERP solution.

The remainder of this paper is organized as follows. A precise definition of the problem, reiterated from Ref. 3, is presented in the next section. For completeness, the ERP method is then briefly reviewed. In the following section we present the new third-order parametrization. As an illustration of its utility, the new method is used to present a new derivation of a widely-used third-order quaternion propagation algorithm. To demonstrate the accuracy and efficiency of the method, we use the same numerical example of Ref. 3 in a simulation study. Conclusions are drawn in the final section.

Problem Statement

The minimal-parameter problem, defined in Ref. 3, is the following. Given the matrix differential equation

$$\dot{V}(t) = W(t)V(t), \qquad V(t_0) = V_0 \tag{2}$$

where $V \in \mathcal{R}^{n,n}$, $W(t)$ is a skew-symmetric matrix for all $t \geq t_0$, and V_0 is orthogonal, the problem is to find 1) a set of $m = n(n-1)/2$ parameters that unambiguously define $V(t)$, 2) the differential

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*Visiting Scientist, Department of Aerospace Engineering; on sabbatical from RAFAEL, Ministry of Defense—Armament Development Authority, P.O. Box 2250, Haifa 31021, Israel.

†Marcella S. Geltman Senior Lecturer, Department of Aerospace Engineering. Senior Member AIAA.

equation satisfied by these parameters, 3) the transformation that maps these parameters into $V(t)$, and 4) an efficient method to solve the differential equation and to compute $V(t)$.

To facilitate a subsequent comparison and for the sake of completeness, the minimal-parameter solution developed by Bar-Itzhack and Markley in Ref. 3 is reviewed next.

Extended Rodrigues Parameters

In this section we briefly review the ERP minimal-parameter method recently developed by Bar-Itzhack and Markley. For detailed presentation and proof, the reader is referred to Ref. 3.

Rodrigues Parameters in Three-Dimensional Space

As previously noted, the three-parameter representation of three-dimensional rotations that was used in Ref. 3 is due to Rodrigues.[6] Denoting these parameters by g_1, g_2, and g_3, the differential equation satisfied by them is[4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1+g_1^2 & g_3+g_1g_2 & -g_2+g_1g_3 \\ -g_3+g_1g_2 & 1+g_2^2 & g_1+g_2g_3 \\ g_2+g_1g_3 & -g_1+g_2g_3 & 1+g_3^2 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{3}$$

where ω_x, ω_y, and ω_z are the three components of the angular velocity vector of the final coordinate system with respect to the initial one, when this vector is resolved in the final system. Using the Rodrigues parameters, the transformation matrix D, which transforms vectors from the initial coordinate system into the rotated one, can be computed as follows:

$$D = \frac{1}{d}\begin{bmatrix} 1+g_1^2-g_2^2-g_3^2 & 2(g_1g_2+g_3) & 2(g_1g_3-g_2) \\ 2(g_1g_2-g_3) & 1-g_1^2+g_2^2-g_3^2 & 2(g_2g_3+g_1) \\ 2(g_1g_3+g_2) & 2(g_2g_3-g_1) & 1-g_1^2-g_2^2+g_3^2 \end{bmatrix} \tag{4}$$

where

$$d = 1+g_1^2+g_2^2+g_3^2 \tag{5}$$

Letting G be the matrix defined as

$$G \triangleq \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix} \tag{6}$$

and defining W as

$$W \triangleq \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{7}$$

yields the following matrix form of Eqs. (3–5):

$$\dot{G} = -\tfrac{1}{2}(I+G)W(I-G) \tag{8}$$

and

$$D = (I-G)(I+G)^{-1} \tag{9}$$

where I is the identity matrix.

Extended Rodrigues Parameters

As previously stated, Bar-Itzhack and Markley extended the three-dimensional Rodrigues parametrization method into the n-dimensional case. The ERP-based solution of Eq. (2) in the n-dimensional case is

$$V(t) = [I-G(t)][I+G(t)]^{-1}V_0 \tag{10}$$

where $G(t)$ is a skew-symmetric matrix satisfying the following differential equation:

$$\dot{G}(t) = -\tfrac{1}{2}[I+G(t)]W(t)[I+G(t)]^T, \qquad G(t_0)=0 \tag{11}$$

Referring to the problem statement, the $n(n-1)/2$ off-diagonal terms of the matrix $G(t)$ are the answer to part 1. The differential equation (11) is the answer to part 2, and Eq. (10), expressing V in terms of the entries of G, is the answer to part 3. To obtain a usable numerical solution that avoids the potential singularity of $I+G$ and the need to invert $I+G$ [see Eq. (10)], an approximate method was suggested in Ref. 3. The method is based on the observation that $V(t)$ can also be expressed using the series

$$V(t) = \left[I + 2\sum_{n=1}^{\infty}(-1)^n G^n(t)\right]V_0 \tag{12}$$

In actual implementation, the degree of approximation is determined by the number of terms used. Thus, a third-order approximation is given by

$$V(t) = (I - 2G(t)\{I - G(t)[I - G(t)]\})V_0 \tag{13}$$

This algorithm provides the answer to part 4 of the minimal-parameter problem.

New Minimal-Parameter Method

This section presents a new minimal set of parameters for the third-order solution of Eq. (1). The parameters are motivated by the Peano–Baker method of solution of linear matrix differential equations,[5] which we use to prove the following theorem.

Theorem 1. Let $V(t) \in \mathcal{R}^{n,n}$ be any time-varying orthogonal matrix that satisfies the matrix differential equation

$$\dot{V}(t) = W(t)V(t) \tag{14a}$$

with

$$V(t_0) = V_0 \tag{14b}$$

where V_0 is orthogonal. Then there exists a unique matrix-valued function $B(t, t_0)$ such that the solution of Eq. (14a) that satisfies the initial condition (14b) can be expressed as

$$V(t) = [I + A(t,t_0) + B(t,t_0)]V_0 \tag{15}$$

where

$$A(t,t_0) \triangleq \int_{t_0}^{t} W(\upsilon)\,\mathrm{d}\upsilon \tag{16}$$

In addition, $B(t, t_0)$ satisfies the algebraic equation

$$\left[I - A(t,t_0) + B^T(t,t_0)\right][I + A(t,t_0) + B(t,t_0)] = I \tag{17}$$

Proof. Dating from 1888, the Peano–Baker method[5] gives the solution of Eqs. (14) subject to the initial condition (14b) as

$$V(t) = \Phi(t,t_0)V_0 \tag{18}$$

where the transition matrix $\Phi(t, t_0)$ is defined by the infinite series

$$\Phi(t,t_0) \triangleq I + A(t,t_0) + \int_{t_0}^{t} W(\upsilon)\int_{t_0}^{\upsilon} W(\upsilon_1)\,\mathrm{d}\upsilon_1\,\mathrm{d}\upsilon + \cdots \tag{19}$$

and $A(t, t_0)$ is defined as in Eq. (16). A straightforward differentiation of Eq. (19) proves that $V(t)$ from Eq. (18) is indeed a solution of Eqs. (14). Now $\Phi(t, t_0)$ can be rewritten as

$$\Phi(t,t_0) = I + A(t,t_0) + B(t,t_0) \tag{20}$$

which is an implied definition of the unique matrix-valued function $B(t, t_0)$. Note that, according to Eq. (19), both $B(t, t_0)$ and its derivative $\dot{B}(t, t_0)$ vanish at $t = t_0$. Since Eqs. (14) are self-adjoint, $\Phi(t, t_0)$ is an orthogonal matrix.[5] Hence

$$\Phi^T(t,t_0)\Phi(t,t_0) = I \tag{21}$$

Rewriting Eq. (21) explicitly in terms of $B(t, t_0)$ and noting that, by its definition (16), $A(t, t_0)$ is a skew-symmetric matrix completes the proof. □

Discussion. Notice that, if Eq. (17) could be explicitly solved for the matrix $B(t, t_0)$ in terms of the entries of $A(t, t_0)$, then this would have implied that the solution of Eqs. (14) can be defined in terms of just the $n(n-1)/2$ off-diagonal terms of $A(t, t_0)$. Stated in another way, this would have meant that the off-diagonal terms of $A(t, t_0)$ constitute a minimal set of parameters that completely determine the solution $V(t)$.

In fact, in the special case where $A(t, t_0)$ and $W(t)$ commute, that is indeed true. As is well known, the exact solution to Eqs. (14) is then

$$V(t) = \Phi(t, t_0)V_0 = \exp[A(t, t_0)]\, V_0 \tag{22}$$

where

$$\exp[A(t, t_0)] = \sum_{i=0}^{\infty} \frac{1}{i!} A^i(t, t_0) \tag{23}$$

implying that the solution $V(t)$ is a function of the off-diagonal terms of $A(t, t_0)$ and, equivalently, that the relation between $B(t, t_0)$ and $A(t, t_0)$ is

$$B(t, t_0) = \sum_{i=2}^{\infty} \frac{1}{i!} A^i(t, t_0) \tag{24}$$

It is easy to verify that, in this special case, the solution $V(t)$ from Eq. (22) is an orthogonal matrix. This follows from the fact that both V_0 and $\exp[A(t, t_0)]$ are orthogonal matrices {the orthogonality of $\exp[A(t, t_0)]$ follows from the skew symmetry of $A(t, t_0)$}.

Unfortunately, in the general case, it can be shown that Eq. (17) does not uniquely define $B(t, t_0)$ in terms of $A(t, t_0)$. However, limiting the scope of the search, we are able to present a minimal parametrization which yields a simple, closed-form third-order solution of Eqs. (14). This parametrization is the subject of the next theorem.

Third-Order Parametrization

Theorem 2. Let $\tilde{V}(t, t_0)$ be the matrix-valued function defined as

$$\tilde{V}(t, t_0) \triangleq \left(I + A(t, t_0) + \frac{1}{2!} A^2(t, t_0) + \frac{1}{3!} A^3(t, t_0) + \frac{t - t_0}{3!} \left\{ A(t, t_0) W_0 - [A(t, t_0) W_0]^T \right\} \right) V_0 \tag{25}$$

where $A(t, t_0)$ is defined in Eq. (16) and $W_0 = W(t_0)$. Then $\tilde{V}(t, t_0)$ is a third-order approximation of the solution $V(t)$.

Proof. $\tilde{V}(t, t_0)$ constitutes a third-order approximation of the solution if all of its derivatives up to order three are equal to the corresponding derivatives of the solution at t_0. The proof consists of a straightforward, direct comparison of the first three derivatives of $V(t)$ and $\tilde{V}(t, t_0)$ at t_0. The derivatives of $\tilde{V}(t, t_0)$ are computed using the differential equation

$$\dot{A}(t, t_0) = W(t), \qquad A(t_0, t_0) = 0 \tag{26}$$

which follows from Eq. (16). □

As follows from Theorem 2, a minimal set of parameters, which defines a third-order solution of Eqs. (14), are the $n(n-1)/2$ off-diagonal terms of the skew-symmetric matrix $A(t, t_0)$. This provides an answer (albeit approximate) to part 1 of the problem statement. Equation (25) provides an answer to part 2 of the problem statement, which requires a mapping of the parameters into the orthogonal matrix V. The parameters are natural, because they are directly related to $W(t)$. Moreover, for the three-dimensional case they have a simple geometric interpretation: they are the angles resulting from a temporal integration of the components of the angular velocity vector $\omega \triangleq [\omega_x \;\; \omega_y \;\; \omega_z]^T$ of the final coordinate system with respect to the initial coordinate system, when that vector is resolved in the final system. Indeed, this interpretation will serve, in the sequel, to provide a new derivation of a third-order algorithm for the integration of the quaternion equation, using the output of an orthogonal triad of rate-integrating gyros.

Equation (26) provides a formal answer to part 2 of the problem statement, which requires a differential equation whose solution defines the parameters. Notice that, since in our case $W(t)$ is a skew-symmetric matrix, only $n(n-1)/2$ integrations are needed to solve Eq. (26). Moreover, the simplicity of Eq. (26) stands out in comparison with Eq. (11), which defines the kinematics of the extended Rodrigues parameters.

Remark 1. In the special case where $A(t, t_0)$ and W_0 commute, Eq. (25) is a third-order approximation of Eq. (22), as could be expected; for in that case Eq. (22) is the exact solution of Eqs. (14). To verify this, notice that since both $A(t, t_0)$ and W_0 are skew-symmetric, we have

$$A(t, t_0)W_0 - [A(t, t_0)W_0]^T = A(t, t_0)W_0 - W_0 A(t, t_0) \tag{27}$$

Hence, when $A(t, t_0)$ and W_0 commute, the last term in $\tilde{V}(t, t_0)$ vanishes, rendering the expression in large parentheses in Eq. (25) equal to the beginning (first four terms) of the series expansion for $\exp[A(t, t_0)]$.

Remark 2. Equation (27) may be used to rewrite Eq. (25) as

$$\tilde{V}(t, t_0) = \left(I + A(t, t_0) + \frac{1}{2!} A^2(t, t_0) + \frac{1}{3!} A^3(t, t_0) + \frac{t - t_0}{3!} [A(t, t_0) W_0 - W_0 A(t, t_0)] \right) V_0 \tag{28}$$

which can be shown to be a third-order solution of any linear matrix differential equation of the form of Eqs. (14), where $W(t)$ is not necessarily skew-symmetric.

Remark 3. For $\tilde{V}(t, t_0)$ to be a valid approximate solution, it should approximate an orthogonal matrix. This issue is addressed in the next theorem.

Theorem 3. $\tilde{V}(t, t_0)$ is a third-order approximation of an orthogonal matrix, in the sense that

$$\tilde{V}(t, t_0)\tilde{V}^T(t, t_0) = I + \mathcal{O}\left[(t - t_0)^4\right] \tag{29}$$

where $\mathcal{O}(x)$ denotes a function of x that has the property that $\mathcal{O}(x)/x$ is bounded as $x \to 0$.

Proof. Let $K(t, t_0)$ denote the matrix consisting of the last term of $\tilde{V}(t, t_0)$ in the large parentheses of Eq. (25), i.e.,

$$K(t, t_0) = \frac{t - t_0}{3!} \{A(t, t_0)W_0 - [A(t, t_0)W_0]^T\} \tag{30}$$

It is obvious that $K(t, t_0)$ is a skew-symmetric matrix. Since $A(t, t_0)$, W_0, and $K(t, t_0)$ are skew-symmetric, then

$$\begin{aligned} \tilde{V}(t, t_0)\tilde{V}^T(t, t_0) &= \left(I + A(t, t_0) + \frac{1}{2!} A^2(t, t_0) + \frac{1}{3!} A^3(t, t_0) + K(t, t_0) \right) \\ &\quad \times \left(I - A(t, t_0) + \frac{1}{2!} A^2(t, t_0) - \frac{1}{3!} A^3(t, t_0) - K(t, t_0) \right) \end{aligned} \tag{31}$$

Using Eq. (26) yields

$$A(t, t_0) = (t - t_0)W_0 + \mathcal{O}\left[(t - t_0)^2\right] \tag{32}$$

hence

$$K(t, t_0) \sim \mathcal{O}\left[(t - t_0)^3\right] \tag{33}$$

Using Eqs. (32) and (33) in Eq. (31) completes the proof. □

Table 1 Computational burden

Method	n_m	n_a	n_s	EOC
New (third-order)	$3n^3 + 5n^2$	$3n^3 - n^2$	$8n^2$	$5n^3 + 7n^2$
ERP (third-order)	$8n^3 + 6.5n^2$	$8n^3 - 8n^2$	$20n^2$	$13.3n^3 + 7.8n^2$
Direct solution (fourth-order)	$4n^3 + 11n^2$	$4n^3 + 4n^2$	$13n^2$	$6.7n^3 + 18n^2$

Numerical Algorithm

To obtain a numerical algorithm as an answer to part 4 of the problem statement, we have to find the entries of $A(t, t_0)$ using a numerical solution of Eq. (26).

Equation (26) is a very simple differential equation, whose solution consists of a direct integration of $W(t)$. A numerical solution using the Simpson quadrature formula is

$$A(t, t_0) = (h/6)\{W(t_0) + 4W[t_0 + (h/2)] + W(t_0 + h)\} + \mathcal{O}(h^5) \tag{34}$$

where $h = t - t_0$. Notice that, in this case, the Simpson formula is equivalent to the fourth-order Runge–Kutta solution of Eq. (26).

The numerical solution of Eq. (14) is obtained by using Eq. (34) in Eq. (25). For given initial conditions $V(t_0) = V_0$ and $W(t_0) = W_0$ and a given integration step h, this yields $V(t_0 + h)$. In the next integration step this procedure is repeated to compute $V(t_0 + 2h)$ from $V(t_0 + h)$, and so on.

Computational Load

To assess the computational efficiency of the new method, its expected computational load is compared in Table 1 with the third-order solution based on the ERP method, Eqs. (11) and (13). The ERP method requires about 1.5 matrix multiplications to compute $\dot{G}$, taking into account the skew symmetry of G. Fourth-order Runge–Kutta integration of Eq. (11) is assumed. The new third-order method, based on Eq. (26), does not require matrix multiplications for the computation of $\dot{A}$. Moreover, simple Simpson quadrature is used (which, as previously mentioned, is equivalent to fourth-order Runge–Kutta integration). Note, however, that several matrix multiplications are needed to calculate $\tilde{V}$.

Since this method is most commonly used in practice, we present in the last row of Table 1 the workload associated with a direct, fourth-order Runge–Kutta integration of Eq. (1). This method requires one matrix multiplication to calculate the derivative $\dot{V}$. Four calculations of the derivatives $\dot{V}$ are required at each time step. It is noted, however, that this method (being fourth-order) is more accurate than the other two methods appearing in Table 1; hence, its computational burden should not be directly compared with the computational requirements of the other methods.

Table 1 shows the numbers of operations for each method, where n_m, n_a, and n_s denote the numbers of multiplications, additions, and substitutions, respectively. Note that each $n \times n$ matrix multiplication requires n^3 scalar multiplications and $n^2(n - 1)$ additions; each integrated variable in a fourth-order Runge–Kutta routine involves 11 multiplications and 8 additions.

The relative efficiency of the three methods is measured by the expression in the last column of Table 1, the equivalent operation count (EOC), defined as the weighted sum

$$\text{EOC} \triangleq n_m + \tfrac{2}{3}n_a + \tfrac{1}{3}n_s \tag{35}$$

The weights in Eq. (35) are based on the number of clock pulses that these operations require in a typical 486-class computer [the EOC does not take into account the additional operations required to generate the matrix $W(t)$, which are identical in all methods].

Table 1 shows that the new method is more efficient than the third-order version of the minimal-parameter ERP method. For a four-dimensional system, the new method saves about 55% of the workload required by the ERP method. For a large n the corresponding expected saving is about 62%. It can also be observed that the new method requires a considerably smaller computational burden than the direct solution (with savings of 40 and 25% for $n = 4$ and large n, respectively).

Remark 4. As previously stated, the direct solution is fourth order; hence it should generally require fewer time steps than either third-order method to achieve a comparable accuracy. In practice, therefore, we can expect lower savings from either third-order method than for the direct, fourth-order solution.

Remark 5. Although a third-order version of the ERP method was used here for the sake of comparison, note that, unlike the new method presented in this paper, the ERP method is not limited to third-order accuracy [see Eq. (12)].

Application: Quaternion Propagation

In this section, we demonstrate the utility of the new method by using it to present a new derivation of a well-known third-order integration algorithm for the quaternion differential equation.

The quaternion is a four-parameter rotation specifier.[4] Popular in navigation and attitude determination applications, its usage eliminates the singularity problem associated with all three-parameter attitude representations (e.g., Euler angles, Rodrigues parameters), although at the price of adding one superfluous parameter. Letting

$$q \triangleq [q_0 \quad q_1 \quad q_2 \quad q_3]^T \tag{36}$$

denote the quaternion vector, the differential equation satisfied by the quaternion elements is

$$\dot{q}(t) = \Omega(t)q(t), \qquad q(t_0) = q_0 \tag{37}$$

where the matrix $\Omega(t)$ is composed of the angular velocity components,

$$\Omega(t) = \tfrac{1}{2}\Psi[\omega(t)] \tag{38}$$

and $\Psi[\cdot]$ is a 4×4 matrix-valued function on $\mathcal{R}^3$ such that

$$\Psi[\omega(t)] \triangleq \begin{bmatrix} 0 & -\omega_x(t) & -\omega_y(t) & -\omega_z(t) \\ \omega_x(t) & 0 & \omega_z(t) & -\omega_y(t) \\ \omega_y(t) & -\omega_z(t) & 0 & \omega_x(t) \\ \omega_z(t) & \omega_y(t) & -\omega_x(t) & 0 \end{bmatrix} \tag{39}$$

Notice that Eq. (37) is a coupled, albeit linear, differential equation.

The problem addressed in this section is that of propagating the attitude quaternion in an approximate closed form from time t to time $t + T$, by processing gyro data. Specifically, we refer herein to a third-order approximation that is based on the assumption that a rate-integrating gyro (RIG) package is used.

Remark 6. The history of the particular third-order scheme referred to herein is interesting in itself. The scheme was apparently first presented in the open literature (without proof) by Grubin.[7] In his paper, which compared three attitude determination schemes based on Euler angles, quaternions, and the direction cosine matrix, Grubin used an empirical modification of a quaternion integration scheme that was originally presented by Edwards[8] (who did not provide any proof or derivation either). Grubin noticed that, when using the scheme suggested in Ref. 8, he obtained better results if the sign of one of the terms in that scheme was reversed, although he did not provide any mathematical explanation of his findings. Roth,[9] who also compared different approximate integration schemes in the context of strapdown attitude determination, stated that Grubin's algorithm (including the sign reversal, which happened to be correct), was rigorously proven by Ben-Dor in an unpublished correspondence. Unfortunately, Roth, too, omitted the complete derivation in his thesis. Finally, Markley and Spence[10] showed how to derive the algorithm by using Taylor series expansion of the quaternion $q(t + T)$ about time t and repeatedly using the quaternion kinematic equation (37).

Remark 7. Although the new minimal-parameter method has been developed for the matrix equation (1), it is also applicable in the case of the vector equation (37), since any column $v_i(t)$, $i \in \{1, \ldots, n\}$ of the orthogonal matrix $V(t)$ in Eq. (1) satisfies the same differential equation, namely

$$\dot{v}_i(t) = W(t)v_i(t) \tag{40}$$

Remark 8. In compliance with standard practice in inertial navigation systems, it is assumed in the sequel that the measuring device is a RIG package, which periodically provides temporal integrals of the angular velocity components along each of the vehicle axes over the sampling interval T. Let the vector

$$\Delta\,\theta(t) \stackrel{\triangle}{=} [\Delta\,\theta_x(t) \quad \Delta\,\theta_y(t) \quad \Delta\,\theta_z(t)]^T \tag{41}$$

denote the vector of RIG outputs at time t; i.e., it is assumed that

$$\Delta\,\theta(t+T) = \int_t^{t+T} \omega(\upsilon)\,\mathrm{d}\upsilon \tag{42}$$

In the sequel, it will be assumed that $q(t)$ has already been determined. The purpose of the integration algorithm to be developed is to compute $q(t+T)$ from the measured gyro outputs $\Delta\,\theta(t)$ and $\Delta\,\theta(t+T)$, using the new third-order parametrization.

Adapted to the present problem, Eq. (28) yields

$$\begin{aligned} q(t+T) &= \left(I + A(t+T,t) + \frac{1}{2!}A^2(t+T,t) + \frac{1}{3!}A^3(t+T,t)\right. \\ &\left. + \frac{T}{3!}[A(t+T,t)\Omega(t) - \Omega(t)A(t+T,t)]\right)q(t) \end{aligned} \tag{43}$$

where the matrix $A(t+T,t)$ is

$$A(t+T,t) = \int_t^{t+T} \Omega(\upsilon)\,\mathrm{d}\upsilon \tag{44}$$

Using Eqs. (38) and (42), we have

$$A(t+T,t) = \tfrac{1}{2}\Psi[\Delta\,\theta(t+T)] \tag{45a}$$

from which it is easy to verify that

$$A^2(t+T,t) = -\tfrac{1}{4}\|\Delta\,\theta(t+T)\|^2 I \tag{45b}$$

$$A^3(t+T,t) = -\tfrac{1}{4}\|\Delta\,\theta(t+T)\|^2 A(t+T,t) \tag{45c}$$

where $\|\cdot\|$ denotes the Euclidean norm, and also

$$\begin{aligned} &A(t+T,t)\Omega(t) - [A(t+T,t)\Omega(t)]^T \\ &= \tfrac{1}{4}\{\Psi[\Delta\,\theta(t+T)]\Psi[\omega(t)] - \Psi[\omega(t)]\Psi[\Delta\,\theta(t+T)]\} \\ &= \tfrac{1}{2}\Psi[\omega(t)\times\Delta\,\theta(t+T)] \end{aligned} \tag{45d}$$

where $\times$ denotes the usual vector product.

In principle, we can now use Eqs. (45) in Eq. (43) to obtain the required third-order quaternion propagational algorithm. Notice, however, that in Eq. (45d) the angular velocity components are explicitly used. Since, as previously noted, it is assumed that a RIG package is utilized, we proceed further by approximating the angular velocity vector using the gyro outputs, as follows:

$$\omega(t) \approx (1/2T)[\Delta\,\theta(t) + \Delta\,\theta(t+T)] \tag{46}$$

Using Eq. (46) in Eq. (45d) yields

$$\begin{aligned} &A(t+T,t)\Omega(t) - [A(t+T,t)\Omega(t)]^T \\ &= (1/4T)\Psi[\Delta\,\theta(t)\times\Delta\,\theta(t+T)] \end{aligned} \tag{47}$$

Substituting Eqs. (45a–45c) and (47) into Eq. (43) finally results in

$$\begin{aligned} q(t+T) &= \Big\{\big(1 - \tfrac{1}{8}\|\Delta\,\theta(t+T)\|^2\big)I \\ &+ \tfrac{1}{2}\big[1 - \tfrac{1}{24}\|\Delta\,\theta(t+T)\|^2\big]\Psi[\Delta\,\theta(t+T)] \\ &+ \tfrac{1}{24}\Psi[\Delta\,\theta(t)\times\Delta\,\theta(t+T)]\Big\}q(t) \end{aligned} \tag{48}$$

which is the sought-for third-order propagation algorithm.

As noted in Refs. 9 and 10, the first two terms on the right-hand side of Eq. (48) represent the change in the quaternion during the time interval T, assuming a constant angular velocity during that interval. These terms follow from a Taylor series expansion of the closed-form, constant-velocity solution of Eq. (37):

$$q(t+T) = \exp\{\tfrac{1}{2}\Psi[\Delta\,\theta(t+T)]\}q(t) \tag{49}$$

The last term on the right-hand side of Eq. (48) may be viewed as a correction term, which represents the change in the quaternion due to the change in the direction of the axis of rotation during the sampling interval.

In passing, we note that the vector product in Eq. (48) can be computed using the cross-product matrix $[\Delta\,\theta(t)\times]$, defined as

$$[\Delta\,\theta(t)\times] \stackrel{\triangle}{=} \begin{bmatrix} 0 & -\Delta\,\theta_z(t) & \Delta\,\theta_y(t) \\ \Delta\,\theta_z(t) & 0 & -\Delta\,\theta_x(t) \\ -\Delta\,\theta_y(t) & \Delta\,\theta_x(t) & 0 \end{bmatrix} \tag{50}$$

Using this definition, the vector product on the right-hand side of Eq. (48) can be computed as

$$\Delta\,\theta(t)\times\Delta\,\theta(t+T) = [\Delta\,\theta(t)\times]\Delta\,\theta(t+T) \tag{51}$$

Numerical Example

In this section we use a numerical example to demonstrate the accuracy and efficiency of the new method. The new method is compared with a third-order version of the ERP method. A reference solution, obtained using a fourth-order direct integration, is used to measure the accuracy of both third-order solutions.

Of the various numerical examples we have used to test the performance of the new method, we have chosen to present here the results of a fourth-order system, which is identical to the system used in Ref. 3. The differential equation is

$$W(t) = \begin{bmatrix} 0 & -0.1 & -1.0 & -7.5 \\ 0.1 & 0 & 3.0 & 0 \\ 1.0 & -3.0 & 0 & -0.9 \\ 7.5 & 0 & 0.9 & 0 \end{bmatrix} \sin(6.28t) \tag{52a}$$

with the initial condition

$$V_0 = I \tag{52b}$$

The initial time is $t_0 = 0$, the final time is $t_f = 0.5$ s, and the integration time step is $h = 0.001$ s. All computations were programmed in Microsoft Fortran 5.1 (a 16-bit implementation of Fortran 77) and implemented using single precision on an Intel 486-powered personal computer. The results are presented for $t = t_f$, which yields the maximal error for all methods (the solution is periodic with cycle time $T = 1$ s, and the maximal error occurs at half the period[3]).

Computed by the new method, the following solution matrix was obtained at $t_f = 0.5$ s:

$$V = \begin{bmatrix} -7.276567\text{E}{-01} & 1.528573\text{E}{-01} & -2.438722\text{E}{-01} & -6.226368\text{E}{-01} \\ 1.021751\text{E}{-02} & 5.837368\text{E}{-01} & 7.919407\text{E}{-01} & -1.788189\text{E}{-01} \\ 1.393533\text{E}{-01} & -7.973786\text{E}{-01} & 5.348135\text{E}{-01} & -2.423716\text{E}{-01} \\ 6.715603\text{E}{-01} & -3.717101\text{E}{-03} & -1.653151\text{E}{-01} & -7.222207\text{E}{-01} \end{bmatrix} \tag{53}$$

Table 2 Error metrics

Method	ε	ν
New	2.248E–06	3.66E–06
ERP	2.345E–06	3.70E–06
Direct solution	——	1.71E–06

Table 3 Computation time

	Actual time		Predicted load	
Method	Time, s	% of direct	EOC	% of direct
New	0.66	57	432	60
ERP	1.86	162	968	135
Direct solution	1.15	100	717	100

This solution was compared with two other solutions.

Using a third-order version of the ERP method, Eqs. (11) and (12), the following solution matrix was obtained:

$$V_{\mathrm{ERP}} = \begin{bmatrix} -7.276566\mathrm{E}{-01} & 1.528573\mathrm{E}{-01} & -2.438722\mathrm{E}{-01} & -6.226367\mathrm{E}{-01} \\ 1.021752\mathrm{E}{-02} & 5.837367\mathrm{E}{-01} & 7.919407\mathrm{E}{-01} & -1.788188\mathrm{E}{-01} \\ 1.393532\mathrm{E}{-01} & -7.973785\mathrm{E}{-01} & 5.348135\mathrm{E}{-01} & -2.423715\mathrm{E}{-01} \\ 6.715605\mathrm{E}{-01} & -3.717093\mathrm{E}{-03} & -1.653151\mathrm{E}{-01} & -7.222205\mathrm{E}{-01} \end{bmatrix} \tag{54}$$

The reference solution, computed via a direct fourth-order Runge–Kutta solution of the matrix differential equation (1), was

$$V_{\mathrm{REF}} = \begin{bmatrix} -7.276558\mathrm{E}{-01} & 1.528571\mathrm{E}{-01} & -2.438721\mathrm{E}{-01} & -6.226372\mathrm{E}{-01} \\ 1.021753\mathrm{E}{-02} & 5.837359\mathrm{E}{-01} & 7.919418\mathrm{E}{-01} & -1.788188\mathrm{E}{-01} \\ 1.393533\mathrm{E}{-01} & -7.973777\mathrm{E}{-01} & 5.348138\mathrm{E}{-01} & -2.423718\mathrm{E}{-01} \\ 6.715592\mathrm{E}{-01} & -3.717092\mathrm{E}{-03} & -1.653150\mathrm{E}{-01} & -7.222208\mathrm{E}{-01} \end{bmatrix} \tag{55}$$

The error associated with each method, relative to the direct integration method, was measured using the metric

$$\varepsilon(V) \triangleq \|V - V_{\mathrm{REF}}\| \tag{56}$$

where $\|\cdot\|$ denotes the Frobenius norm.

The deviation from orthogonality of the solution in the various methods was measured using the metric

$$\nu(V) \triangleq \|V V^T - I\| \tag{57}$$

The resulting metric values are shown in Table 2. As can be observed, the errors of the two minimal-parameter methods relative to the direct solution are small and similar to each other. The measures of deviation from orthogonality for the two minimal-parameter methods are also small and similar (though more than twice as large as the corresponding measure for the direct method).

Although the numerical results are similar for the three methods, the new method is significantly more efficient. The actual computation times are shown in Table 3. The computational time saving of the new method is very close to that expected from the EOC measure (see previous section): it requires less than 60% of the direct method. The ERP method is significantly slower (notice that the EOC measure provided only an approximate prediction of its efficiency).

Conclusions

A new third-order minimal-parameter method has been presented for the solution of the orthogonal matrix differential equation. This equation plays an important role in various navigation- and estimation-related problems, e.g., transformation between rotated coordinate systems, or the solution of the matrix Riccati equation. The new method is motivated by the Peano–Baker method of solution of linear matrix differential equations. The parameters, and their corresponding differential equation, are very simple and natural to the problem. Moreover, for the three-dimensional case of transformation matrices, the parameters have a simple geometric interpretation, being the angles resulting from a temporal integration of the three angular velocity components of the rotating coordinate system with respect to the reference coordinate system.

The new method was used to obtain a new, simple derivation of a known third-order algorithm for the numerical propagation of the attitude quaternion, which is widely used in inertial navigation systems utilizing rate-integrating gyros.

The accuracy and high numerical efficiency of the new method were demonstrated via a numerical example taken from the literature. Both algorithm operation count and the numerical example have demonstrated that the use of the new method can result in substantial computation time savings in cases where third-order accuracy is sufficient.

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