Weiss–Weinstein Lower Bounds for Markovian Systems. Part 1: Theory

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***Abstract*—Being essentially free from regularity conditions, the Weiss–Weinstein estimation error lower bound can be applied to a larger class of systems than the well-known Cramér–Rao lower bound. Thus, this bound is of special interest in applications involving hybrid systems, i.e., systems with both continuously and discretely distributed parameters, which can represent, in practice, fault-prone systems. However, the requirement to know explicitly the joint distribution of the estimated parameters with all the measurements makes the application of the Weiss–Weinstein lower bound to Markovian dynamic systems cumbersome. A sequential algorithm for the computation of the Cramér–Rao lower bound for such systems has been recently reported in the literature. Along with the marginal state distribution, the algorithm makes use of the transitional distribution of the Markovian state process and the distribution of the measurements at each time step conditioned on the appropriate states, both easily obtainable from the system equations. A similar technique is employed herein to develop sequential Weiss–Weinstein lower bounds for a class of Markovian dynamic systems. In particular, it is shown that in systems satisfying the Cramér–Rao lower bound regularity conditions, the sequential Weiss–Weinstein lower bound derived herein reduces, for a judicious choice of its parameters, to the sequential Cramér–Rao lower bound.**

***Index Terms*—Dynamic Markovian systems, estimation error lower bound.**

I. Introduction

THE problem of *a priori* assessment of estimation errors arises each time a suitable system architecture is required in order to achieve some prescribed estimation performance. A natural way of performing such an assessment is to investigate the behavior of the estimation error covariance matrix produced by the appropriate minimum mean-square error (MMSE) filter. Unfortunately, a closed-form optimal filter is unavailable for the majority of practical systems, calling for the use of alternative estimation error measures, e.g., lower bounds on the estimation error covariance matrix.

The most popular lower bound is the well-known Cramér–Rao lower bound (CRLB). This bound is presented in [1, p. 84] in the context of Bayesian estimation of static random parameters. In that formulation, also known as the Van Trees version of the CRLB, the underlying static random system is assumed to satisfy the so-called CRLB regularity conditions [1, p. 72], that is, absolute integrability of the first two derivatives of all related probability density functions (pdf's). Later, [2], [3] provided a CRLB derivation under less restrictive requirements, but even these weaker conditions state, among others, that all associated pdf's must be continuously differentiable [3].

The first derivation of a sequential CRLB version applicable to discrete-time dynamic system filtering, the problem addressed in this work, was presented in [4] and then extended in [5]–[7]. Recently, the most general form of sequential CRLB for discrete-time nonlinear systems was presented in [8]. Together with the original static form of the CRLB, these results served as a basis for a large number of applications [9]–[13].

Unfortunately, the requirement to satisfy the CRLB regularity conditions, even in their weakest form, rules out the use of the CRLB in many practical applications. A suitable example are hybrid systems, i.e., systems with both continuously and discretely distributed parameters [14], which may be used to model, e.g., maneuvering targets [15] or a fault-prone behavior [16, p. 177]. Two bounds for such systems, which have been recently presented by the authors, are [17] and [18].

There is still another, more powerful bound, known in the literature as the Weiss–Weinstein lower bound (WWLB) [3]. Both the CRLB and the Bobrovsky–Zakai lower bound are special cases of the WWLB [19]. Being essentially free from regularity conditions, the WWLB can be applied to a very large class of estimation problems, including estimation of discretely distributed parameters. However, in order to use the WWLB, one has to know the joint distribution of all the estimated parameters with all the measurements. Unfortunately, the computation of this distribution is very cumbersome in dynamic systems described by Markov processes, where only the transitional distribution of the states and the distribution of the measurements at each time step conditioned on the appropriate states are directly available. Therefore, direct application of the WWLB to dynamic systems is impractical.

Based on the authors' conference paper [20], the present work derives a sequential form of the WWLB via an extension of the technique proposed in [8]. It is shown that, under certain conditions, the sequential form exists. Directly using the transitional state distribution, the measurement conditional distribution, and the marginal state distribution, renders the resulting class of WWLB's practical for Markovian dynamic systems. It is also shown that the sequential CRLB [8] is a special case of the new

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sequential lower bound. Moreover, the companion paper [21] presents an application of the sequential version of the WWLB to several classes of fault-prone systems and relates it to other existing sequential lower bounds.

The remainder of this paper is organized as follows. The standard WWLB is presented in Section II. In Section III, the Markovian dynamic system of interest is defined, and some basic assumptions are made and discussed. The paper's main result, namely the sequential version of the WWLB, is derived in Section IV. The relation of this bound to the sequential CRLB of [8] is presented in Section V. Concluding remarks are offered in the last section. For improved readability, several auxiliary results are presented in Appendices A and B. For presentation clarity, the notational convention of [1] is adopted, according to which lower-case and upper-case letters are used to denote random variables and their realizations, respectively.

II. Weiss–Weinstein Lower Bound

Let $x \in \mathbb{R}^n$ be a random vector of parameters and let $y \in \mathbb{R}^m$ be a corresponding measurement vector. Then, according to [3], for any matrix

$$H = [\, h_1 \quad h_2 \quad \cdots \quad h_n \,] \in \mathbb{R}^{n \times n} \tag{1}$$

with columns $h_i \in \mathbb{R}^n$, $i \in \{1, \ldots, n\}$, any estimator $\hat{x}(y)$, and any set of numbers $\{s_1, s_2, \ldots, s_n\}$ such that $s_i \in [0,1]$, the Weiss–Weinstein lower bound on the estimation error covariance matrix of x is

$$E\left[(x - \hat{x}(y))(x - \hat{x}(y))^T\right] \geqslant H G^{-1} H^T \tag{2}$$

where the (i,j) element of G is given as shown by equation (3) at the bottom of the page, and the likelihood ratio $L(Y; X_1, X_2)$ is defined as

$$L(Y; X_1, X_2) \triangleq \frac{p_{y,x}(Y, X_1)}{p_{y,x}(Y, X_2)}. \tag{4}$$

The integration in the mathematical expectations is carried out over the support of the joint pdf $p_{y,x}$, denoted by $\operatorname{supp} p_{y,x}$. Notice that the matrix G is symmetric.

Remark 2.1: The matrix H and the set of numbers $\{s_1, s_2, \ldots, s_n\}$ are arbitrary. Thus, (2) presents a family of estimation error lower bounds. Moreover, these quantities can be used as tuning parameters to tighten the bound. As reported in [3], the choice $s_i = 0.5$, $i = 1, 2, \ldots, n$ often maximizes the WWLB.

III. Definitions and Basic Assumptions

A. *Markovian Dynamic System*

Consider a dynamic system, characterized by a Markovian state process $\{z_k\}_{k=0}^{\infty}$ that is measured through a measurement process $\{y_k\}_{k=1}^{\infty}$, where $z_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^m$. The time histories of the states and the measurements are denoted by

$$\mathcal{Z}_k \triangleq \left[z_0^T, z_1^T, \ldots, z_k^T \right]^T \tag{5a}$$

$$\mathcal{Y}_k \triangleq \left[y_1^T, y_2^T, \ldots, y_k^T \right]^T \tag{5b}$$

and their realizations are denoted by

$$\Xi_k \triangleq \left[Z_0^T, Z_1^T, \ldots, Z_k^T \right]^T \tag{6a}$$

$$\Upsilon_k \triangleq \left[Y_1^T, Y_2^T, \ldots, Y_k^T \right]^T. \tag{6b}$$

The joint distribution of $\mathcal{Z}_k$ and $\mathcal{Y}_k$ is given by

$$p_{\mathcal{Z}_k, \mathcal{Y}_k}(\Xi_k, \Upsilon_k) = \prod_{l=0}^{k} p_{y_l|z_l}(Y_l \mid Z_l)\, p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) \tag{7}$$

where

$$p_{z_0|z_{-1}}(Z_0 \mid Z_{-1}) = p_{z_0}(Z_0) \tag{8a}$$

$$p_{y_0|z_0}(Y_0 \mid Z_0) = p_{y_0}(Y_0). \tag{8b}$$

Finally, for some time instant l and some vector $h(l) \in \mathbb{R}^n$, define

$$\mathcal{A}_{h(l)} \triangleq \big\{ Z_{l-1} \text{ such that } \{ Z_{l-1} \in \operatorname{supp} p_{z_{l-1}} \} \wedge \{ Z_{l-1} + h(l) \in \operatorname{supp} p_{z_{l-1}} \} \big\} \tag{9}$$

and define $\alpha_l(h(l), Z_{l-1})$ as shown by equation (10) at the bottom of the next page. Then, the sequential bound to be developed in the sequel applies to systems satisfying the following fundamental assumption.

Assumption 3.1: For every time instant l there exists $h(l) \in \mathbb{R}^n$ satisfying

$$\mathcal{A}_{h(l)} \neq \emptyset \tag{11}$$

$$G_{ij} = \frac{E\left[\left(L^{s_i}(y; x + h_i, x) - L^{1-s_i}(y; x - h_i, x)\right)\left(L^{s_j}(y; x + h_j, x) - L^{1-s_j}(y; x - h_j, x)\right)\right]}{E\left[L^{s_i}(y; x + h_i, x)\right] E\left[L^{s_j}(y; x + h_j, x)\right]} \tag{3}$$

such that

$$\alpha_l(h(l), Z_{l-1}) = \alpha_l(h(l)) \quad \forall\, Z_{l-1} \in \mathcal{A}_{h(l)} \tag{12}$$

Remark 3.1: In systems satisfying Assumption 3.1, $\alpha_l(h)$ is an even function of h, that is, $\alpha_l(h) = \alpha_l(-h)$.

Remark 3.2: The quantity $\alpha_l(h(l), Z_{l-1})$ is related to the sensitivity of the state transitional distribution $p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1})$ to changes in the value of the conditioning state vector z_{l-1}. Assumption 3.1 states that this sensitivity is uniform over all values of z_{l-1}.

Assumption 3.1 might seem overly restrictive. However, as illustrated in the sequel, it is valid for a wide range of dynamic systems.

1) Class 3.1 (Systems With Linear Dynamics): Let the state process satisfy

$$z_l = \Phi z_{l-1} + u_l + w_l \tag{13}$$

where the process noise $\{w_l\}_{l=1}^{\infty}$ is a white sequence with pdf $p_{w_l}(W_l)$, and $\{u_l\}_{l=1}^{\infty}$ be a known deterministic input sequence. In this case

$$p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) = p_{w_l}(Z_l - \Phi Z_{l-1} - u_l) \tag{14}$$

so that equation (15) at the bottom of the page holds, thus satisfying Assumption 3.1. Notice that only the dynamics equation is required to be linear. The measurement equation may be nonlinear, and the system noises are not restricted to be Gaussian.

2) Class 3.2 (Bernoulli Markov Chain): If $\{z_k\}_{k=1}^{\infty}$ is a Bernoulli random sequence then

$$\operatorname{supp} p_{z_{l-1}} = \{0, 1\}. \tag{16}$$

In this case, the admissible values of h are ± 1. Moreover, $\mathcal{A}_1 = \{0\}$ and $\mathcal{A}_{-1} = \{1\}$, i.e., there exists only one value of Z_{l-1} corresponding to each value of h. Therefore, Assumption 3.1 is trivially satisfied with

$$\alpha_l(h, Z_{l-1}) = \int_{-\infty}^{+\infty} \sqrt{p_{z_l|z_{l-1}}(Z_l \mid 0)\, p_{z_l|z_{l-1}}(Z_l \mid 1)}\, dZ_l. \tag{17}$$

3) Class 3.3 (Systems Satisfying the CRLB Regularity Conditions With $h \to 0$): Letting $h \to 0$ yields

$$\begin{aligned}\alpha_l(h, Z_{l-1}) &= \int_{-\infty}^{+\infty} \Big(p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) \\ &\quad + \frac{1}{2} \nabla_{Z_{l-1}}^T p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) h \\ &\quad + p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) R(h, Z_l, Z_{l-1}) \Big) dZ_l\end{aligned} \tag{18}$$

where the residual term, $R(h, Z_l, Z_{l-1})$, is $o(h)$, i.e., it goes to zero faster than h, and

$$\nabla_\zeta \triangleq \left[\frac{\partial}{\partial \zeta_1} \quad \frac{\partial}{\partial \zeta_2} \quad \cdots \quad \frac{\partial}{\partial \zeta_n} \right]^T, \quad \forall\, \zeta \in \mathbb{R}^n \tag{19}$$

is the gradient operator. If the CRLB regularity conditions are satisfied, the order of integration and differentiation may be interchanged, yielding

$$\alpha_l(h, Z_{l-1}) = 1 + \int_{-\infty}^{+\infty} p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1}) R(h, Z_l, Z_{l-1})\, dZ_l. \tag{20}$$

The integral in (20) is $o(h)$. Thus, for sufficiently small values of h, $\alpha_l(h, Z_{l-1})$ does not depend on Z_{l-1}.

B. Lower Bound Restrictions and Notation

The derivation procedure of the sequential version of the WWLB is based on the idea presented in [8]. The lower bound for the state vector z_{k+1} is derived from the standard WWLB applied to the entire history $\mathcal{Z}_{k+1}$. For this lower bound, the following notation is used. Let the matrices $H^{(k+1)}$ and $G^{(k+1)}$ be the matrices defined in (1) and (3), respectively, when calculated for the histories $\mathcal{Z}_{k+1}$ and $\mathcal{Y}_{k+1}$. To permit sequential calculation, the matrix $H^{(k+1)}$ is chosen to be block-diagonal

$$H^{(k+1)} = \begin{bmatrix} H(0,0) & & \\ & \ddots & \\ & & H(k+1,k+1) \end{bmatrix} \tag{21}$$

$$\alpha_l(h(l), Z_{l-1}) \triangleq \int_{-\infty}^{+\infty} \sqrt{p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1} + h(l))\, p_{z_l|z_{l-1}}(Z_l \mid Z_{l-1})}\, dZ_l \tag{10}$$

$$\begin{aligned}\alpha_l(h, Z_{l-1}) &= \int_{-\infty}^{+\infty} \sqrt{p_{w_l}(Z_l - \Phi Z_{l-1} - \Phi h - u_l)\, p_{w_l}(Z_l - \Phi Z_{l-1} - u_l)}\, dZ_l \\ &= \int_{-\infty}^{+\infty} \sqrt{p_{w_l}(W - \Phi h)\, p_{w_l}(W)}\, dW,\end{aligned} \tag{15}$$

where the diagonal block

$$H(l,l) = \begin{bmatrix} h_l^{(1)} & h_l^{(2)} & \dots & h_l^{(n)} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{22}$$

corresponds to the state vector z_l. Let h_j (without the superindex) denote the jth column of the matrix $H^{(k+1)}$, in accordance with (1). Notice that, because of the block diagonal structure of $H^{(k+1)}$

$$h_{n(k+1)+i} = \Big[\underbrace{0,\dots,0}_{n(k+1)}, h_{k+1}^{(i)\,T}\Big]^T \tag{23a}$$

$$h_{nl+i} = \Big[\underbrace{0,\dots,0}_{nl}, h_l^{(i)T}, \underbrace{0,\dots,0}_{n(k+1-l)}\Big]^T. \tag{23b}$$

In addition, let $s_i = 0.5$, $i = 1, 2, \dots, n$. The following partition of the matrix $G^{(k+1)}$ is introduced

$$G^{(k+1)} = \begin{bmatrix} G^{(k+1)}(0,0) & \dots & G^{(k+1)}(0,k+1) \\ \vdots & \ddots & \vdots \\ G^{(k+1)}(k+1,0) & \dots & G^{(k+1)}(k+1,k+1) \end{bmatrix} \tag{24}$$

where $G^{(k+1)}(\xi,\eta) \in \mathbb{R}^{n\times n}$. Notice that due to the symmetry of $G^{(k+1)}$

$$G^{(k+1)}(\eta,\xi) = G^{(k+1)}(\xi,\eta)^T. \tag{25}$$

Finally, let

$$\begin{aligned} &L_l\left(Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right) \\ &\quad \triangleq \frac{p_{y_l|z_l}\left(Y_l \mid Z_l^{(1)}\right) p_{z_l|z_{l-1}}\left(Z_l^{(1)} \mid Z_{l-1}\right)}{p_{y_l|z_l}\left(Y_l \mid Z_l^{(2)}\right) p_{z_l|z_{l-1}}\left(Z_l^{(2)} \mid Z_{l-1}\right)} \end{aligned} \tag{26a}$$

and

$$\begin{aligned} &K_l\left(Z_{l+1}, Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right) \\ &\quad \triangleq \frac{p_{z_{l+1}|z_l}\left(Z_{l+1} \mid Z_l^{(1)}\right)}{p_{z_{l+1}|z_l}\left(Z_{l+1} \mid Z_l^{(2)}\right)} L_l\left(Y_l; Z_l^{(1)}, Z_l^{(2)}; Z_{l-1}\right). \end{aligned} \tag{26b}$$

IV. Main Result

In this section, the sequential version of the WWLB is derived. For presentation clarity, only the main highlights of the derivation are presented herein. A complete account of the derivation details can be found in Appendix A.

The derivation procedure hinges on the following fact. For a general Markovian dynamic system, the matrices $G^{(k+1)}$ and $G^{(k)}$ have the following form (see Corollary A.1):

$$G^{(k+1)} = \begin{bmatrix} A & B \\ B^T & G^{(k+1)}(k+1,k+1) \end{bmatrix} \tag{27a}$$

$$G^{(k)} = \begin{bmatrix} \Psi & \Theta \\ \Theta^T & G^{(k)}(k,k) \end{bmatrix} \tag{27b}$$

where

$$A = \begin{bmatrix} \Psi & \Omega \\ \Omega^T & G^{(k+1)}(k,k) \end{bmatrix} \tag{28a}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ G^{(k+1)}(k,k+1) \end{bmatrix} \tag{28b}$$

$$\Theta = \begin{bmatrix} 0 \\ \vdots \\ G^{(k)}(k-1,k) \end{bmatrix} \tag{28c}$$

$$\Omega = \begin{bmatrix} 0 \\ \vdots \\ G^{(k+1)}(k-1,k) \end{bmatrix} \tag{28d}$$

and, in general, $\Omega \neq \Theta$. However, when Assumption 3.1 holds, Lemma A.6 states that

$$G^{(k+1)}(k,k-1) = G^{(k)}(k,k-1) \tag{29}$$

so that

$$\Omega = \Theta. \tag{30}$$

Now, let J_l denote the inverse of the (l,l) block of $G^{(l)^{-1}}$. The main result is now stated in the following Theorem.

Theorem 4.1: Consider a Markovian dynamic system. Let $H(l,l)$, $l = 1, 2, \dots, k+1$ be a set of matrices, composed from such columns for which Assumption 3.1 holds. Then, the corresponding WWLB is given by

$$\begin{aligned} &E\left[(z_{k+1} - \hat{z}_{k+1|k+1})(z_{k+1} - \hat{z}_{k+1|k+1})^T\right] \\ &\qquad \geqslant H(k+1,k+1) J_{k+1}^{-1} H(k+1,k+1)^T \end{aligned} \tag{31}$$

where the matrix J_{k+1} is computed sequentially as

$$\begin{aligned} J_{k+1} = &\, G^{(k+1)}(k+1,k+1) \\ &- G^{(k+1)}(k+1,k) \\ &\times \left(J_k + G^{(k+1)}(k,k) - G^{(k)}(k,k)\right)^{-1} \\ &\times G^{(k+1)}(k,k+1) \end{aligned} \tag{32a}$$

$$J_0 = G^{(0)}(0,0) \tag{32b}$$

and the (i,j) entries of the matrices $G^{(k+1)}(\xi,\eta)$ are computed using (33a)–(33c), as shown at the bottom of the next page.

Proof: The result (31) follows from the definition of J_{k+1} and the particular selection of the matrix $H^{(k+1)}$ in (21). In addition, the fact

$$G^{(0)} = G^{(0)}(0,0) \tag{34}$$

yields (32b).

The recursion (32a) is proved as follows. Let C denote the (k,k) block of A^{-1}. It follows from (27), (28), and (30) that

$$\begin{aligned} J_{k+1} &= G^{(k+1)}(k+1,k+1) - B^T A^{-1} B \\ &= G^{(k+1)}(k+1,k+1) - G^{(k+1)}(k+1,k) \\ &\quad \times C G^{(k+1)}(k,k+1) \qquad (35) \end{aligned}$$

$$C = \left(G^{(k+1)}(k,k) - \Theta^T \Psi \Theta\right)^{-1} \qquad (36)$$

$$J_k = G^{(k)}(k,k) - \Theta^T \Psi \Theta. \qquad (37)$$

Using (37), (36) can be rewritten as

$$C = \left(J_k + G^{(k+1)}(k,k) - G^{(k)}(k,k)\right)^{-1} \qquad (38)$$

and (32a) is obtained upon substituting (38) into (35). Finally, (33) follows from Lemma A.3. ■

Remark 4.1: Notice that the selection of the matrices $H(l,l)$ may be not unique. Therefore, the result given by Theorem 4.1 constitutes, in essence, a wide class of WWLBs. Moreover, the matrices $H(l,l)$ can be regarded as tuning parameters, which can be used to tighten the bound.

V. Relation to the Cramér–Rao Lower Bound

Constituting a special case of the WWLB, the CRLB can be recovered from it via a limiting procedure provided that the well-known regularity conditions are satisfied [19]. It will be shown now that the sequential CRLB, derived in [8], is a special case of the sequential version of the WWLB derived in

$$\begin{aligned} &G^{(k+1)}(k+1,k+1)_{ij} \\ &= E\Bigg[\Bigg(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\Bigg) \\ &\quad \times \Bigg(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(j)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(j)}, z_{k+1}; z_k\right)}\Bigg)\Bigg] \\ &\quad \times \frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right] E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(j)}, z_{k+1}; z_k\right)}\right]} \end{aligned} \qquad (33a)$$

$$\begin{aligned} &G^{(k+1)}(k+1,k)_{ij} = G^{(k+1)}(k,k+1)_{ij}^T \\ &= E\Bigg[\Bigg(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\Bigg) \\ &\quad \times \Bigg(\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(j)}, z_k; z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1}, y_k; z_k-h_k^{(j)}, z_k; z_{k-1}\right)}\Bigg)\Bigg] \\ &\quad \times \frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right] E\left[\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(j)}, z_k; z_{k-1}\right)}\right]} \end{aligned} \qquad (33b)$$

$$\begin{aligned} &G^{(k+1)}(k,k)_{ij} \\ &= E\Bigg[\Bigg(\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(i)}, z_k; z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1}, y_k; z_k-h_k^{(i)}, z_k; z_{k-1}\right)}\Bigg) \\ &\quad \times \Bigg(\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(j)}, z_k; z_{k-1}\right)} - \sqrt{K_k\left(z_{k+1}, y_k; z_k-h_k^{(j)}, z_k; z_{k-1}\right)}\Bigg)\Bigg] \\ &\quad \times \frac{1}{E\left[\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(i)}, z_k; z_{k-1}\right)}\right] E\left[\sqrt{K_k\left(z_{k+1}, y_k; z_k+h_k^{(j)}, z_k; z_{k-1}\right)}\right]} \end{aligned} \qquad (33c)$$

Section IV. To facilitate the development, the sequential CRLB is presented next.

Theorem 5.1 (Tichavský, Muravchik, and Nehorai, 1998): Suppose that the joint distribution given in (7) satisfies the CRLB regularity conditions [1, p. 72]. Define

$$D_k^{11} = -E\left[\Delta_{z_k}^{z_k} \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)\right] \quad (39a)$$

$$D_k^{12} = -E\left[\Delta_{z_k}^{z_{k+1}} \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)\right] = {D_k^{21}}^T \quad (39b)$$

$$D_k^{22} = -E\left[\Delta_{z_{k+1}}^{z_{k+1}} \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)\right] - E\left[\Delta_{z_{k+1}}^{z_{k+1}} \ln p_{y_{k+1}|z_{k+1}}(y_{k+1} \mid z_{k+1})\right] \quad (39c)$$

where the operator Δ_ξ^η is defined, using the gradient operator, as

$$\Delta_\xi^\eta \triangleq \nabla_\xi \nabla_\eta^T. \quad (40)$$

Then, the sequential CRLB for this system is given by [8]

$$E\left[(z_{k+1} - \hat{z}_{k+1|k+1})(z_{k+1} - \hat{z}_{k+1|k+1})^T\right] \geqslant {J_{k+1}^*}^{-1} \quad (41)$$

where the Fisher information sub-matrix J_{k+1}^* is computed via the following recursion

$$J_{k+1}^* = D_k^{22} - D_k^{21}\left(J_k^* + D_k^{11}\right)^{-1} D_k^{12} \quad (42a)$$

$$J_0^* = -E\left[\Delta_{z_0}^{z_0} \ln p_{z_0}(z_0)\right] \quad (42b)$$

Theorem 5.2: If the joint distribution given in (7) satisfies the CRLB regularity conditions [1, p. 72], then the sequential CRLB given in Theorem 5.1 follows from the sequential version of the WWLB presented in Theorem 4.1, for the following selection of the matrix $H^{(k+1)}$:

$$H^{(k+1)} = \varepsilon I_{n(k+2)\times n(k+2)}, \quad \varepsilon \to 0. \quad (43)$$

Proof: For conciseness, only the underlying idea of the proof is presented herein. The proof details are deferred to Appendix B.

For the particular selection of the matrix $H^{(k+1)}$ as given in (43), the system under consideration belongs to Class 3.3 presented in Section III. Therefore, Assumption 3.1 is satisfied and Theorem 4.1 holds. Using a Taylor expansion in (33) yields the following expressions for the matrices: $G^{(\cdot)}(\cdot,\cdot)$

$$G^{(k+1)}(k+1,k+1) = \varepsilon^2 D_k^{22} + o(\varepsilon^2) \quad (44a)$$

$$G^{(k+1)}(k,k+1) = G^{(k+1)}(k+1,k)^T = \varepsilon^2 D_k^{12} + o(\varepsilon^2) \quad (44b)$$

$$G^{(k+1)}(k,k) - G^{(k)}(k,k) = \varepsilon^2 D_k^{11} + o(\varepsilon^2) \quad (44c)$$

where the matrices D_k^{ij} are defined in (39). Substituting (44) into (32a) yields the following recursion for J_{k+1}:

$$\begin{aligned} J_{k+1} &= \varepsilon^2 D_k^{22} - \varepsilon^2\left[D_k^{12} + o(\varepsilon^2)\right]^T \\ &\quad \times \left[J_k + \varepsilon^2 D_k^{11} + o(\varepsilon^2)\right]^{-1} \\ &\quad \times \varepsilon^2\left[D_k^{12} + o(\varepsilon^2)\right] + o\left(\varepsilon^2\right) \\ &= \varepsilon^2\left[D_k^{22} - D_k^{21}\left(\frac{1}{\varepsilon^2}J_k + D_k^{11}\right)^{-1} \times D_k^{12} + \frac{o(\varepsilon^2)}{\varepsilon^2}\right]. \end{aligned} \quad (45)$$

Let J_{k+1}^* denote the inverse of the WWLB in the case where $\varepsilon \to 0$. According to (31), for the special selection of the matrix $H^{(k+1)}$ in (43), this inverse is given by

$$\begin{aligned} J_{k+1}^* &= \lim_{\varepsilon\to 0}\left[H(k+1,k+1)J_{k+1}^{-1}H(k+1,k+1)^T\right]^{-1} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}J_{k+1}. \end{aligned} \quad (46)$$

Together with (45), (46) yields the recursion (42a). In addition, for the particular selection of the matrix $H^{(k+1)}$

$$G^{(0)}(0,0) = -\varepsilon^2 E\left[\Delta_{z_0}^{z_0} \ln p_{z_0}(z_0)\right] + o(\varepsilon^2) \quad (47)$$

which yields (42b). ∎

VI. Conclusion

A sequential computational algorithm is presented for the Weiss–Weinstein estimation error lower bound. Making use of the state transitional distribution, the conditional distribution of the measurements given the appropriate states and the marginal state distribution, the new sequential form renders the application of the Weiss–Weinstein bound feasible in a wide class of dynamical systems, including hybrid systems. For a certain choice of its parameters, this lower bound is shown to reduce to the recently presented sequential Cramér–Rao lower bound, if the corresponding system satisfies the Cramér–Rao lower bound regularity conditions. Several applications of the new sequential form of the Weiss–Weinstein bound to filtering problems in fault-prone hybrid systems are presented by the authors in a companion paper.

Appendix A
Lower Bound Derivation Details

This Appendix provides the derivation details of the sequential version of the WWLB for a general Markovian system. The details are summarized in the following Lemmas and Corollaries.

Lemma A.1: Consider the state and measurement time histories as defined in (5), with joint distribution as given by (7). Then, using the notation (26), the likelihood ratio (4) can be calculated as in (A.1a) and (A.1b), as shown at the bottom of the next page, where the vectors h_j are defined in (23).

Proof: To prove (A.1a), notice equation (A.2), shown at the bottom of the page.

Equation (A.1b) is proved similarly. ■

Lemma A.2: Equation (A.3a) and (A.3b), shown at the bottom of the page, hold.

Proof: The result (A.3a) is proven as shown in equation (A.4) at the bottom of the page. The result (A.3b) is proven similarly. ■

Lemma A.3: Let $G^{(k+1)}(\xi,\eta)_{ij}$ denote the (i,j) element in the (ξ,η) block of the matrix $G^{(k+1)}$, and let $\xi,\eta \leqslant k$. Then, equations (A.5a)–(A.5c) at the bottom of the next page hold.

Proof: It follows from the partitions (21) and (24) and the structural relations (23) that

$$G^{(k+1)}(\xi,\eta)_{ij} = G^{(k+1)}_{n\xi+i,n\eta+j}. \tag{A.6}$$

The Lemma follows from using (A.1) in (3). ■

Lemma A.4:

$$G^{(k+1)}(\xi,\eta) = 0, \quad \forall\, \eta \leqslant \xi - 2 \tag{A.7}$$

Proof: Let $\xi = k+1$. due to the Markov property, given z_k, the product terms inside the expectation in the numerator

$$L(\Upsilon_{k+1};\Xi_{k+1}+h_{nk+i},\Xi_{k+1}) = L_{k+1}\left(Y_{k+1};Z_{k+1}+h^{(i)}_{k+1},Z_{k+1};Z_k\right) \tag{A.1a}$$

$$L(\Upsilon_{k+1};\Xi_{k+1}+h_{n(l-1)+i},\Xi_{k+1}) = K_l\left(Z_{l+1},Y_l;Z_l+h^{(i)}_l,Z_l;Z_{l-1}\right) \tag{A.1b}$$

$$\begin{aligned}
&L(\Upsilon_{k+1};\Xi_{k+1}+h_{n(k+1)+i},\Xi_{k+1})\\
&\quad= \frac{p_{\mathcal{Z}_{k+1},\mathcal{Y}_{k+1}}(\Xi_{k+1}+h_{n(k+1)+i},\Upsilon_{k+1})}{p_{\mathcal{Z}_{k+1},\mathcal{Y}_{k+1}}(\Xi_{k+1},\Upsilon_{k+1})}\\
&\quad= \frac{p_{y_{k+1}|z_{k+1}}\left(Y_{k+1}\mid Z_{k+1}+h^{(i)}_{k+1}\right)p_{z_{k+1}|z_k}\left(Z_{k+1}+h^{(i)}_{k+1}\mid Z_k\right)}{p_{y_{k+1}|z_{k+1}}(Z_{k+1}\mid Z_{k+1})\,p_{z_{k+1}|z_k}(Z_{k+1}\mid Z_k)}\\
&\qquad\times\frac{\prod_{l=0}^{k}p_{y_l|z_l}(Y_l\mid Z_l)p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}{\prod_{l=0}^{k}p_{y_l|z_l}(Y_l\mid Z_l)p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}\\
&\quad= L_{k+1}\left(Y_{k+1};Z_{k+1}+h^{(i)}_{k+1},Z_{k+1};Z_k\right)
\end{aligned} \tag{A.2}$$

$$E\left[\sqrt{L_l(y_l;z_l+h,z_l;z_{l-1})}-\sqrt{L_l(y_l;z_l-h,z_l;z_{l-1})}\mid z_{l-1}\right]=0 \tag{A.3a}$$

$$E\left[\sqrt{K_l(z_{l+1},y_l;z_l+h,z_l;z_{l-1})}-\sqrt{K_l(z_{l+1},y_l;z_l-h,z_l;z_{l-1})}\mid z_{l-1}\right]=0 \tag{A.3b}$$

$$\begin{aligned}
&E\left[\sqrt{L_l(y_l;z_l+h,z_l;z_{l-1})}-\sqrt{L_l(y_l;z_l-h,z_l;z_{l-1})}\mid z_{l-1}\right]\\
&\quad=\int_{-\infty}^{+\infty}\left[\sqrt{\frac{p_{y_l|z_l}(Y_l\mid Z_l+h)\,p_{z_l|z_{l-1}}(Z_l+h\mid Z_{l-1})}{p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}}-\sqrt{\frac{p_{y_l|z_l}(Y_l\mid Z_l-h)\,p_{z_l|z_{l-1}}(Z_l-h\mid Z_{l-1})}{p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}}\right]\\
&\qquad\times p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})\,dY_l dZ_l\\
&\quad=\int_{-\infty}^{+\infty}\sqrt{p_{y_l|z_l}(Y_l\mid Z_l+h)\,p_{z_l|z_{l-1}}(Z_l+h\mid Z_{l-1})\,p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}dY_l dZ_l\\
&\qquad-\int_{-\infty}^{+\infty}\sqrt{p_{y_l|z_l}(Y_l\mid Z_l-h)\,p_{z_l|z_{l-1}}(Z_l-h\mid Z_{l-1})\,p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}dY_l dZ_l\\
&\quad=\int_{-\infty}^{+\infty}\sqrt{p_{y_l|z_l}(Y_l\mid Z_l+h)\,p_{z_l|z_{l-1}}(Z_l+h\mid Z_{l-1})\,p_{y_l|z_l}(Y_l\mid Z_l)\,p_{z_l|z_{l-1}}(Z_l\mid Z_{l-1})}dY_l dZ_l\\
&\qquad-\int_{-\infty}^{+\infty}\sqrt{p_{y_l|z_l}(Y_l\mid \xi)\,p_{z_l|z_{l-1}}(\xi\mid Z_{l-1})\,p_{y_l|z_l}(Y_l\mid \xi+h)\,p_{z_l|z_{l-1}}(\xi+h\mid Z_{l-1})}dY_l d\xi=0
\end{aligned} \tag{A.4}$$

of $G^{(k+1)}(k+1,\eta)_{ij}$ are independent. Therefore, using the smoothing property of the conditional expectation, the numerator can be expressed as shown by (A.8) at the bottom of the next page.

According to Lemma A.2, the first conditional expectation is zero, which proves the Lemma. The proof for $\xi < k+1$ is similar. ■

Lemma A.5: Let $\xi, \eta \leqslant k-1$. Then

$$G^{(k+1)}(\xi,\eta) = G^{(k)}(\xi,\eta) \tag{A.9}$$

where $G^{(k)}$ is the Weiss–Weinstein lower bound G-matrix computed for the state history until time k (see Theorem 4.1). In addition,

$$G^{(k+1)}(k,\eta) = G^{(k)}(k,\eta), \quad \forall\, \eta \leqslant k-2. \tag{A.10}$$

Proof: According to (A.5c), the expressions in $G^{(k+1)}(\xi,\eta)$ for $\xi,\eta \leqslant k-1$ do not include either y_{k+1} or z_{k+1}. Therefore, they are identical in both $G^{(k+1)}$ and $G^{(k)}$, thus establishing (A.9). In addition, according to Lemma A.4

$$G^{(k+1)}(k,\eta) = 0 = G^{(k)}(k,\eta), \quad \forall\, \eta \leqslant k-2 \tag{A.11}$$

which yields (A.10). ■

Corollary A.1: In a general Markovian system the matrices $G^{(k+1)}$ and $G^{(k)}$ can be written as

$$G^{(k+1)} = \begin{bmatrix} A & B \\ B^T & G^{(k+1)}(k+1,k+1) \end{bmatrix} \tag{A.12a}$$

$$G^{(k)} = \begin{bmatrix} \Psi & \Theta \\ \Theta^T & G^{(k)}(k,k) \end{bmatrix} \tag{A.12b}$$

$$\begin{aligned} &G^{(k+1)}(k+1,k+1)_{ij} \\ &= E\left[\left(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right)\right. \\ &\quad \left.\times\left(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(j)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(j)}, z_{k+1}; z_k\right)}\right)\right] \\ &\quad \times \frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right] E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(j)}, z_{k+1}; z_k\right)}\right]} \end{aligned} \tag{A.5a}$$

$$\begin{aligned} &G^{(k+1)}(k+1,\eta)_{ij} = G^{(k+1)}(\eta,k+1)_{ij}^T \\ &= E\left[\left(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}-h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right)\right. \\ &\quad \left.\times\left(\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta+h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)} - \sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta-h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)}\right)\right] \\ &\quad \times \frac{1}{E\left[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1}+h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\right] E\left[\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta+h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)}\right]} \end{aligned} \tag{A.5b}$$

$$\begin{aligned} &G^{(k+1)}(\xi,\eta)_{ij} \\ &= E\left[\left(\sqrt{K_\xi\left(z_{\xi+1}, y_\xi; z_\xi+h_\xi^{(i)}, z_\xi; z_{\xi-1}\right)} - \sqrt{K_\xi\left(z_{\xi+1}, y_\xi; z_\xi-h_\xi^{(i)}, z_\xi; z_{\xi-1}\right)}\right)\right. \\ &\quad \left.\times\left(\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta+h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)} - \sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta-h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)}\right)\right] \\ &\quad \times \frac{1}{E\left[\sqrt{K_\xi\left(z_{\xi+1}, y_\xi; z_\xi+h_\xi^{(i)}, z_\xi; z_{\xi-1}\right)}\right] E\left[\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta+h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)}\right]} \end{aligned} \tag{A.5c}$$

where

$$A = \begin{bmatrix} \Psi & \Omega \\ \Omega^T & G^{(k+1)}(k,k) \end{bmatrix} \tag{A.13a}$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ G^{(k+1)}(k,k+1) \end{bmatrix} \tag{A.13b}$$

$$\Theta = \begin{bmatrix} 0 \\ \vdots \\ G^{(k)}(k-1,k) \end{bmatrix} \tag{A.13c}$$

$$\Omega = \begin{bmatrix} 0 \\ \vdots \\ G^{(k+1)}(k-1,k) \end{bmatrix} \tag{A.13d}$$

and, in general, $\Omega \neq \Theta$.

Whereas the results obtained so far do not require Assumption 3.1, the next Lemma addresses systems satisfying this Assumption.

Lemma A.6: Under Assumption 3.1

$$G^{(k+1)}(k,k-1) = G^{(k)}(k,k-1). \tag{A.14}$$

Proof: According to (A.5c) the numerator of $G^{(k+1)}(k,k-1)_{ij}$ is given by

$$\begin{aligned} E\Bigg[\Bigg(&\sqrt{K_k\left(z_{k+1}, y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)} \\ &- \sqrt{K_k\left(z_{k+1}, y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)}\Bigg) \\ &\times \Bigg(\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)} \\ &- \sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} - h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\Bigg)\Bigg]. \end{aligned} \tag{A.15}$$

For notational simplicity, define

$$\begin{aligned} &f(z_k, y_{k-1}, z_{k-1}, z_{k-2}) \\ &\triangleq \sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)} \\ &\quad - \sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} - h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}. \end{aligned} \tag{A.16}$$

Using this notation, the numerator of $G^{(k+1)}(k,k-1)_{ij}$ becomes (A.17), as shown at the bottom of the next page. Now, using the smoothing property of the conditional expectation yields (A.18), as shown at the bottom of the next page. Using Assumption 3.1 yields (A.19), shown at the bottom of the next page. Also

$$p_{\mathcal{Z}_k}\left(Z_k + h_k^{(i)}, Z_{k-1}, \ldots, Z_0\right) = 0 \quad \forall\, Z_k + h_k^{(i)} \notin \operatorname{supp} p_{z_k} \tag{A.20}$$

yielding

$$L_k\left(Y_k; Z_k + h_k^{(i)}, Z_k; Z_{k-1}\right) = 0 \quad \forall\, Z_k + h_k^{(i)} \notin \operatorname{supp} p_{z_k}. \tag{A.21}$$

Hence,

$$\begin{aligned} &E\Bigg[\sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)} \\ &\qquad \times f(z_k, y_{k-1}, z_{k-1}, z_{k-2}) \mid z_k = Z_k\Bigg] = 0 \\ &\forall\, Z_k + h_k^{(i)} \notin \operatorname{supp} p_{z_k}. \end{aligned} \tag{A.22}$$

$$\begin{aligned} E\Bigg[&\Bigg(\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1} + h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1} - h_{k+1}^{(i)}, z_{k+1}; z_k\right)}\Bigg) \\ &\times \Bigg(\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta + h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)} - \sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta - h_\eta^{(j)}, z_\eta; z_{\mathrm{H}-1}\right)}\Bigg)\Bigg] \\ = E\Bigg\{&E\Bigg[\sqrt{L_{k+1}\left(y_{k+1}; z_{k+1} + h_{k+1}^{(i)}, z_{k+1}; z_k\right)} - \sqrt{L_{k+1}\left(y_{k+1}; z_{k+1} - h_{k+1}^{(i)}, z_{k+1}; z_k\right)} \mid z_k\Bigg] \\ &\times E\Bigg[\sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta + h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)} - \sqrt{K_\eta\left(z_{\eta+1}, y_\eta; z_\eta - h_\eta^{(j)}, z_\eta; z_{\eta-1}\right)} \mid z_k\Bigg]\Bigg\} \end{aligned} \tag{A.8}$$

Equations (A.19) and (A.22) yield (A.23), as shown at the bottom of the page. Equation (A.24) at the bottom of the next page is derived similarly. Substituting (A.23) and (A.24) into (A.17) yields (A.25), as shown at the bottom of the next page, where the

$$\begin{aligned}
&E\left[\left(\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k(y_k;z_k+h_k^{(i)},z_k;z_{k-1})}\right.\right.\\
&\qquad\left.\left.-\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k-h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k(y_k;z_k-h_k^{(i)},z_k;z_{k-1})}\right)f(z_k,y_{k-1},z_{k-1},z_{k-2})\right]\\
&\qquad=E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k(y_k;z_k+h_k^{(i)},z_k;z_{k-1})}f(z_k,y_{k-1},z_{k-1},z_{k-2})\right.\\
&\qquad\quad\left.-\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k-h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k(y_k;z_k-h_k^{(i)},z_k;z_{k-1})}f(z_k,y_{k-1},z_{k-1},z_{k-2})\right]
\end{aligned}\tag{A.17}$$

$$\begin{aligned}
&E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k(y_k;z_k+h_k^{(i)},z_k;z_{k-1})}f(z_k,y_{k-1},z_{k-1},z_{k-2})\right]\\
&\qquad=E\left[E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\mid z_k\right]\right.\\
&\qquad\quad\left.\times E\left[\sqrt{L_k(y_k;z_k+h_k^{(i)},z_k;z_{k-1})}f(z_k,y_{k-1},z_{k-1},z_{k-2})\mid z_k\right]\right]
\end{aligned}\tag{A.18}$$

$$\begin{aligned}
&E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\mid z_k=Z_k\right]\\
&\qquad=\int_{-\infty}^{+\infty}\sqrt{p_{z_{k+1}|z_k}\left(Z_{k+1}\mid Z_k+h_k^{(i)}\right)p_{z_{k+1}|z_k}\left(Z_{k+1}\mid Z_k\right)}dZ_{k+1}\\
&\qquad=\alpha_{k+1}(h_k^{(i)})\quad\forall\, Z_k+h_k^{(i)}\in\operatorname{supp}p_{z_k}
\end{aligned}\tag{A.19}$$

$$\begin{aligned}
&E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k+h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1}\mid z_k\right)}}\sqrt{L_k\left(y_k;z_k+h_k^{(i)},z_k;z_{k-1}\right)}f(z_k,y_{k-1},z_{k-1},z_{k-2})\right]\\
&\qquad=\alpha_{k+1}\left(h_k^{(i)}\right)E\left[\sqrt{L_k\left(y_k;z_k+h_k^{(i)},z_k;z_{k-1}\right)}f(z_k,y_{k-1},z_{k-1},z_{k-2})\right]
\end{aligned}\tag{A.23}$$

last expectation is the numerator of $G^{(k)}(k,k-1)_{ij}$. Similarly, according to (A.5c), the denominator of $G^{(k+1)}(k,k-1)_{ij}$ becomes

$$
\begin{aligned}
&E\left[\sqrt{K_k\left(z_{k+1}, y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)}\right] \\
&\quad \times E\left[\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right] \\
&= E\left[E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k + h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k\right)}} \right.\right. \\
&\qquad \left.\left. \times \sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)} \mid z_k\right]\right] \\
&\quad \times E\left[\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right] \\
&= \alpha_{k+1}\left(h_k^{(i)}\right) E\left[\sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)}\right] \\
&\quad \times E\left[\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right].
\end{aligned} \tag{A.26}
$$

Combining (A.25) and(A.26) yields (A.27), as shown at the bottom of the next page. ■

Appendix B
Details of Proof of Theorem 5.2

This Appendix provides the derivation details of (44).

$$
\begin{aligned}
&E\left[\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k - h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k\right)}}\sqrt{L_k\left(y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)} f(z_k, y_{k-1}, z_{k-1}, z_{k-2})\right] \\
&\quad = \alpha_{k+1}\left(h_k^{(i)}\right) E\left[\sqrt{L_k\left(y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)} f(z_k, y_{k-1}, z_{k-1}, z_{k-2})\right].
\end{aligned} \tag{A.24}
$$

$$
\begin{aligned}
&E\left[\left(\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k + h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k\right)}}\sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)}\right.\right. \\
&\qquad \left.\left. -\sqrt{\frac{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k - h_k^{(i)}\right)}{p_{z_{k+1}|z_k}\left(z_{k+1} \mid z_k\right)}}\sqrt{L_k\left(y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)}\right) f(z_k, y_{k-1}, z_{k-1}, z_{k-2})\right] \\
&= \alpha_{k+1}\left(h_k^{(i)}\right) E\left[\sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)} f(z_k, y_{k-1}, z_{k-1}, z_{k-2})\right. \\
&\qquad \left. -\sqrt{L_k\left(y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)} f(z_k, y_{k-1}, z_{k-1}, z_{k-2})\right] \\
&= \alpha_{k+1}\left(h_k^{(i)}\right) E\left[\left(\sqrt{L_k(y_k; z_k + h_k^{(i)}, z_k; z_{k-1})} - \sqrt{L_k\left(y_k; z_k - h_k^{(i)}, z_k; z_{k-1}\right)}\right)\right. \\
&\qquad \times \left(\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right. \\
&\qquad \left.\left. -\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} - h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right)\right]
\end{aligned} \tag{A.25}
$$

Using a Taylor expansion in (33) yields (B.1), as shown at the bottom of the page, where $z_l^{(i)}$ is the ith component of the state vector z_l. Similarly

$$\begin{aligned}\sqrt{K_l(z_{l+1}, y_l; z_l + h_l^{(i)}, z_l; z_{l-1})} &= 1 + \frac{\varepsilon}{2}\left[\frac{\partial \ln p_{z_{l+1}|z_l}(z_{l+1} \mid z_l)}{\partial z_l^{(i)}} + \frac{\partial \ln p_{y_l|z_l}(y_l \mid z_l)}{\partial z_l^{(i)}} \right. \\ &\quad \left. + \frac{\partial \ln p_{z_l|z_{l-1}}(z_l \mid z_{l-1})}{\partial z_l^{(i)}}\right] + o(\varepsilon). \qquad \text{(B.2)}\end{aligned}$$

Notice that, by the smoothing property of the conditional expectation

$$\begin{aligned}E\left[\frac{\partial \ln p_{y_l|z_l}(y_l \mid z_l)}{\partial z_l^{(i)}}\right] &= E\left[E\left[\frac{\partial \ln p_{y_l|z_l}(y_l \mid z_l)}{\partial z_l^{(i)}} \mid z_l\right]\right] \\ &= E\left[\frac{\partial}{\partial z_l^{(i)}} \int_{-\infty}^{+\infty} p_{y_l|z_l}(Y_l \mid z_l)\, dY_l\right] = 0. \qquad \text{(B.3)}\end{aligned}$$

The change of order between integration and differentiation in (B.3) is justified by noting that the system satisfies the CRLB regularity conditions. Similarly

$$E\left[\frac{\partial \ln p_{z_{l+1}|z_l}(z_{l+1} \mid z_l)}{\partial z_l^{(i)}}\right] = 0. \qquad \text{(B.4)}$$

Also, using again the smoothing property of the conditional expectation but conditioning upon all components of z_l but $z_l^{(i)}$ yields

$$E\left[\frac{\partial \ln p_{z_l|z_{l-1}}(z_l \mid z_{l-1})}{\partial z_l^{(i)}}\right] = 0. \qquad \text{(B.5)}$$

Now, taking mathematical expectation of both sides of (B.1) and (B.2) and substituting (B.3), (B.4), and (B.5) yields

$$E\left[\sqrt{L_l(y_l; z_l + h_l^{(i)}, z_l; z_{l-1})}\right] = 1 + o(\varepsilon) \qquad \text{(B.6a)}$$

$$E\left[\sqrt{K_l(z_{l+1}, y_l; z_l + h_l^{(i)}, z_l; z_{l-1})}\right] = 1 + o(\varepsilon) \qquad \text{(B.6b)}$$

which means that the denominators in the expressions $G^{(k+1)}(\xi, \eta)_{ij}$ given in (33) are all $1 + o(\varepsilon)$. To compute the numerators in the expressions for $G^{(k+1)}(\xi, \eta)_{ij}$ notice that, according to (B.1) and (B.2)

$$\begin{aligned}&\sqrt{L_l(y_l; z_l + h_l^{(i)}, z_l; z_{l-1})} - \sqrt{L_l(y_l; z_l - h_l^{(i)}, z_l; z_{l-1})} \\ &\quad = \varepsilon\left[\frac{\partial \ln p_{y_l|z_l}(y_l \mid z_l)}{\partial z_l^{(i)}} + \frac{\partial \ln p_{z_l|z_{l-1}}(z_l \mid z_{l-1})}{\partial z_l^{(i)}}\right] + o(\varepsilon) \qquad \text{(B.7)}\end{aligned}$$

$$\begin{aligned}G^{(k+1)}(k, k-1)_{ij} &= \alpha_{k+1}\left(h_k^{(i)}\right) \\ &\times E\left[\left(\sqrt{L_k(y_k; z_k + h_k^{(i)}, z_k; z_{k-1})} - \sqrt{L_k(y_k; z_k - h_k^{(i)}, z_k; z_{k-1})}\right)\right. \\ &\times \left.\left(\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)} - \sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} - h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right)\right] \\ &\times \frac{1}{\alpha_{k+1}(h_k^{(i)}) E\left[\sqrt{L_k\left(y_k; z_k + h_k^{(i)}, z_k; z_{k-1}\right)}\right] E\left[\sqrt{K_{k-1}\left(z_k, y_{k-1}; z_{k-1} + h_{k-1}^{(j)}, z_{k-1}; z_{k-2}\right)}\right]} \\ &= G^{(k)}(k, k-1)_{ij} \qquad \text{(A.27)}\end{aligned}$$

$$\begin{aligned}\sqrt{L_l(y_l; z_l + h_l^{(i)}, z_l; z_{l-1})} &= \sqrt{\frac{p_{y_l|z_l}\left(y_l \mid z_l + h_l^{(i)}\right) p_{z_l|z_{l-1}}\left(z_l + h_l^{(i)} \mid z_{l-1}\right)}{p_{y_l|z_l}(y_l \mid z_l)\, p_{z_l|z_{l-1}}(z_l \mid z_{l-1})}} \\ &= 1 + \frac{\varepsilon}{2}\left[\frac{\partial \ln p_{y_l|z_l}(y_l \mid z_l)}{\partial z_l^{(i)}} + \frac{\partial \ln p_{z_l|z_{l-1}}(z_l \mid z_{l-1})}{\partial z_l^{(i)}}\right] + o(\varepsilon) \qquad \text{(B.1)}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{K_l(z_{l+1},y_l;z_l+h_l^{(i)},z_l;z_{l-1})}\\&\quad-\sqrt{K_l(z_{l+1},y_l;z_l-h_l^{(i)},z_l;z_{l-1})}\\&\quad=\varepsilon\left[\frac{\partial\ln p_{z_{l+1}|z_l}(z_{l+1}\mid z_l)}{\partial z_l^{(i)}}+\frac{\partial\ln p_{y_l|z_l}(y_l\mid z_l)}{\partial z_l^{(i)}}\right.\\&\qquad\left.+\frac{\partial\ln p_{z_l|z_{l-1}}(z_l\mid z_{l-1})}{\partial z_l^{(i)}}\right]+o(\varepsilon).\end{aligned}\tag{B.8}$$

Substituting (B.7) into (33a) yields

$$\begin{aligned}&G^{(k+1)}(k+1,k+1)_{ij}\\&\quad=\frac{1}{1+o(\varepsilon)}\\&\quad\times E\left[\varepsilon^2\left(\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(i)}}\right.\right.\\&\qquad\left.+\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(i)}}\right)\\&\qquad\times\left(\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right.\\&\qquad\left.\left.+\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(j)}}\right)+o(\varepsilon^2)\right].\end{aligned}\tag{B.9}$$

But

$$\begin{aligned}&E\left[\frac{\partial\ln p_{y_l|z_l}(y_l\mid z_l)}{\partial z_l^{(i)}}\frac{\partial\ln p_{z_l|z_{l-1}}(z_l\mid z_{l-1})}{\partial z_l^{(j)}}\right]\\&\quad=E\left[E\left[\frac{\partial\ln p_{y_l|z_l}(y_l\mid z_l)}{\partial z_l^{(i)}}\mid z_l\right]\right.\\&\qquad\left.\times E\left[\frac{\partial\ln p_{z_l|z_{l-1}}(z_l\mid z_{l-1})}{\partial z_l^{(j)}}\mid z_l\right]\right]=0\end{aligned}\tag{B.10}$$

yielding (B.11) at the bottom of the page. Similarly, substituting (B.7) and (B.8) into (33b) gives

$$\begin{aligned}&G^{(k+1)}(k,k+1)_{ij}\\&\quad=E\left[\left(\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_k^{(i)}}+\frac{\partial\ln p_{y_k|z_k}(y_k\mid z_k)}{\partial z_k^{(i)}}\right.\right.\\&\qquad\left.+\frac{\partial\ln p_{z_k|z_{k-1}}(z_k\mid z_{k-1})}{\partial z_k^{(i)}}\right)\\&\qquad\times\left(\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right.\\&\qquad\left.\left.+\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(j)}}\right)\right]\varepsilon^2+o(\varepsilon^2).\end{aligned}\tag{B.12}$$

Using the smoothing property of the conditional expectation it can be shown that

$$E\left[\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_k^{(i)}}\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right]=0\tag{B.13a}$$

$$E\left[\frac{\partial\ln p_{y_k|z_k}(y_k\mid z_k)}{\partial z_k^{(i)}}\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right]=0\tag{B.13b}$$

$$E\left[\frac{\partial\ln p_{z_k|z_{k-1}}(z_k\mid z_{k-1})}{\partial z_k^{(i)}}\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right]=0\tag{B.13c}$$

$$E\left[\frac{\partial\ln p_{y_k|z_k}(y_k\mid z_k)}{\partial z_k^{(i)}}\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(j)}}\right]=0\tag{B.13d}$$

$$E\left[\frac{\partial\ln p_{z_k|z_{k-1}}(z_k\mid z_{k-1})}{\partial z_k^{(i)}}\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(j)}}\right]=0.\tag{B.13e}$$

Hence

$$G^{(k+1)}(k,k+1)_{ij}$$

$$\begin{aligned}&G^{(k+1)}(k+1,k+1)_{ij}\\&\quad=\frac{1}{1+o(\varepsilon)}\left(\varepsilon^2E\left[\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(i)}}\frac{\partial\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(j)}}\right]\right.\\&\qquad\left.+\varepsilon^2E\left[\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(i)}}\frac{\partial\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(j)}}\right]+o(\varepsilon^2)\right)\\&\quad=-\varepsilon^2E\left[\frac{\partial^2\ln p_{y_{k+1}|z_{k+1}}(y_{k+1}\mid z_{k+1})}{\partial z_{k+1}^{(i)}\partial z_{k+1}^{(j)}}\right]-\varepsilon^2E\left[\frac{\partial^2\ln p_{z_{k+1}|z_k}(z_{k+1}\mid z_k)}{\partial z_{k+1}^{(i)}\partial z_{k+1}^{(j)}}\right]+o(\varepsilon^2).\end{aligned}\tag{B.11}$$

$$
\begin{aligned}
&= \varepsilon^2 E\left[\frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)}}\right.\\
&\qquad \left.\times \frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_{k+1}^{(j)}}\right] + o(\varepsilon^2)\\
&= -\varepsilon^2 E\left[\frac{\partial^2 \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)} \partial z_{k+1}^{(j)}}\right] + o(\varepsilon^2).
\end{aligned} \tag{B.14}
$$

In addition, substituting (B.8) into (33c) yields

$$
\begin{aligned}
&G^{(k+1)}(k,k)_{ij}\\
&= \varepsilon^2 E\left[\left(\frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)}} + \frac{\partial \ln p_{y_k|z_k}(y_k \mid z_k)}{\partial z_k^{(i)}}\right.\right.\\
&\qquad \left. + \frac{\partial \ln p_{z_k|z_{k-1}}(z_k \mid z_{k-1})}{\partial z_k^{(i)}}\right)\\
&\qquad \times\left(\frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(j)}} + \frac{\partial \ln p_{y_k|z_k}(y_k \mid z_k)}{\partial z_k^{(j)}}\right.\\
&\qquad \left.\left. + \frac{\partial \ln p_{z_k|z_{k-1}}(z_k \mid z_{k-1})}{\partial z_k^{(j)}}\right)\right] + o(\varepsilon^2).
\end{aligned} \tag{B.15}
$$

Again, the smoothing property of the conditional expectation gives

$$
E\left[\frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)}} \frac{\partial \ln p_{y_k|z_k}(y_k \mid z_k)}{\partial z_k^{(j)}}\right] = 0 \tag{B.16a}
$$

$$
E\left[\frac{\partial \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)}} \frac{\partial \ln p_{z_k|z_{k-1}}(z_k \mid z_{k-1})}{\partial z_k^{(j)}}\right] = 0 \tag{B.16b}
$$

so that using (B.10) yields

$$
\begin{aligned}
&G^{(k+1)}(k,k)_{ij}\\
&= -\varepsilon^2 E\left[\frac{\partial^2 \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)} \partial z_k^{(j)}}\right]\\
&\quad - \varepsilon^2 E\left[\frac{\partial^2 \ln p_{y_k|z_k}(y_k \mid z_k)}{\partial z_k^{(i)} \partial z_k^{(j)}}\right]\\
&\quad - \varepsilon^2 E\left[\frac{\partial^2 \ln p_{z_k|z_{k-1}}(z_k \mid z_{k-1})}{\partial z_k^{(i)} \partial z_k^{(j)}}\right] + o(\varepsilon^2).
\end{aligned} \tag{B.17}
$$

Finally, according to (B.11)

$$
\begin{aligned}
&G^{(k)}(k,k)_{ij}\\
&= -\varepsilon^2 E\left[\frac{\partial^2 \ln p_{y_k|z_k}(y_k \mid z_k)}{\partial z_k^{(i)} \partial z_k^{(j)}}\right]\\
&\quad - \varepsilon^2 E\left[\frac{\partial^2 \ln p_{z_k|z_{k-1}}(z_k \mid z_{k-1})}{\partial z_k^{(i)} \partial z_k^{(j)}}\right] + o(\varepsilon^2)
\end{aligned} \tag{B.18}
$$

so that

$$
\begin{aligned}
&G^{(k+1)}(k,k)_{ij}\\
&\quad = G^{(k)}(k,k)_{ij}\\
&\qquad - \varepsilon^2 E\left[\frac{\partial^2 \ln p_{z_{k+1}|z_k}(z_{k+1} \mid z_k)}{\partial z_k^{(i)} \partial z_k^{(j)}}\right] + o(\varepsilon^2).
\end{aligned} \tag{B.19}
$$

Using the definitions (39) one can observe that

$$
G^{(k+1)}(k+1,k+1) = \varepsilon^2 D_k^{22} + o(\varepsilon^2) \tag{B.20a}
$$

$$
\begin{aligned}
G^{(k+1)}(k,k+1) &= G^{(k+1)}(k+1,k)^T\\
&= \varepsilon^2 D_k^{12} + o(\varepsilon^2)
\end{aligned} \tag{B.20b}
$$

$$
G^{(k+1)}(k,k) - G^{(k)}(k,k) = \varepsilon^2 D_k^{11} + o(\varepsilon^2) \tag{B.20c}
$$

which are (44).

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