A Cramér–Rao-Type Estimation Lower Bound for Systems With Measurement Faults

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***Abstract*—A Cramér–Rao-type lower bound is presented for systems with measurements prone to discretely-distributed faults, which are a class of hybrid systems. Lower bounds for both the state and the Markovian interruption variables (fault indicators) of the system are derived, using the recently presented sequential version of the Cramér–Rao lower bound (CRLB) for general nonlinear systems. Because of the hybrid nature of the systems addressed, the CRLB cannot be directly applied due to violation of its associated regularity conditions. To facilitate the calculation of the lower bound, the hybrid system is first approximated by a system in which the discrete distribution of the fault indicators is replaced by an approximating continuous one. The lower bound is then obtained via a limiting process applied to the approximating system. The results presented herein facilitate a relatively simple calculation of a nontrivial lower bound for the state vector of systems with fault-prone measurements. The CRLB-type lower bound for the interruption process variables turns out to be trivially zero, however, a nontrivial, non-CRLB-type bound for these variables has been recently presented elsewhere by the authors. The utility and applicability of the proposed lower bound are demonstrated via a numerical example involving a simple global positioning system (GPS)-aided navigation system, where the GPS measurements are fault-prone due to their sensitivity to multipath errors.**

***Index Terms*—Estimation error lower bound, fault detection and isolation, hybrid systems.**

I. Introduction

MODERN multisensor applications, such as navigation and target tracking systems, require the fusion of data acquired by a large number of different sensors. In many situations these sensors might be subjected to faults, either due to internal malfunctions, or because of external interferences. These sensor faults are usually manifested as a sudden addition of noise (white or colored) to the sensor measurements, or even interruption of the output signal. Thus, global positioning system (GPS) jamming and spoofing signals take, in general, the form of colored noises and appear suddenly whenever they are activated [1]. Magnetometer faults, caused by magnetic fields generated by spacecraft electronics and electromagnetic torquing coils, usually take the form of biases [2, p. 251] and appear whenever the corresponding current starts. Rate gyro faults, caused by input accelerations if the gyro gimbals are not perfectly balanced, are also usually modeled as biases [2, p. 198] and appear whenever the spacecraft accelerates. In view of present day systems' high accuracy requirements, the problem of fault tolerant filtering in multisensor systems is of major importance.

Characterized by sudden structural changes, fault-prone system behavior is usually modeled and analyzed using the framework of hybrid systems [3, p. 177]. The total state of these systems comprises two kinds of parameters: the continuously distributed parameters, usually referred to as the system states, and a Markovian switching parameter, which takes values in a finite set and is referred to, in general, as the system mode. Considering fault-prone systems, one of the switching parameter values corresponds to the nominal system operation, whereas the others represent various fault conditions [1], [4]. In systems with independent fault sources and fault-free dynamics, such as GPS-aided inertial navigation systems, the aforementioned model can be simplified: The faults caused by different sources can be modeled as separate Markovian Bernoulli random processes, where "1" stands for a fault situation and "0" stands for no fault situation. Since the state vector is free of faults, these fault indicators affect only the system measurements.

Because of its importance, the problem of stability and control of hybrid systems has drawn considerable research efforts over the past several decades. Depending on the particular engineering problem, a varying level of certainty in the hybrid models was assumed in various works. Thus, in [3] and [5]–[7] the stability and control of hybrid systems was investigated under the assumption that both the state and the mode variables are known. In other works (see, e.g., [8] and [9]), a partial-knowledge structure was adopted: only the state vector was assumed to be known, whereas the system mode was not directly observed and, consequently, had to be estimated.

In systems with fault-prone sensors, direct access to both the state and the mode parameters is naturally unavailable, so that the problem of their simultaneous estimation is of prime importance. It is well-known that the mean square optimal filtering algorithm for hybrid systems, that provides the estimates of the state vector and the switching parameters, requires infinite computation resources [10]. Therefore, a variety of suboptimal estimation techniques was proposed [11]–[16]. Since the estimates of the state vector and the fault indicators are suboptimal, it is of particular interest to obtain some measure of their efficiency. The natural means for this purpose is the comparison to a lower bound on the estimation error.

The most popular lower bound is the well-known Cramér–Rao lower bound (CRLB). This bound is presented in [17, p. 84] in the context of Bayesian estimation of static random parameters. In that formulation, also known as the Van Trees version of the CRLB, the underlying static random

Manuscript received September 16, 2003; revised November 3, 2004 and May 1, 2005. Recommended by Associate Editor H. Wang.

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Digital Object Identifier 10.1109/TAC.2005.854579

system is assumed to satisfy the so-called CRLB regularity conditions [17, p. 72], that is, absolute integrability of the first two derivatives of all related probability density functions (pdfs). Later, [18] and [19] provided a CRLB derivation under less restrictive requirements, but even these weaker conditions state, among others, that all related pdfs must be continuously differentiable [19].

The first derivation of a sequential CRLB version applicable to discrete-time dynamic system filtering, the problem addressed in this paper, was done in [20] and then extended in [21]–[23]. Recently, the most general form of sequential CRLB for discrete-time nonlinear systems was presented in [24]. Together with the original static form of the CRLB, these results served as a basis for a large number of applications [25]–[29].

Unfortunately, albeit being a very useful tool for systems with continuously distributed parameters, the CRLB cannot be directly calculated for both the state and the mode variables of a hybrid system. The reason lies in the fact that the system of interest must satisfy the CRLB regularity conditions. Clearly, this is not the case in hybrid systems in general and in systems with fault-prone measurements in particular, since the mode variables, or fault indicators, are discrete Markovian sequences. The CRLB can be applied directly only to the state vector (see, e.g., [29]), which is continuously distributed and, in most cases, satisfies the regularity conditions. However, the application of the sequential form of the CRLB [24] is based on a special requirement regarding the structure of the system measurements, which, as will be shown later, is not satisfied by a general class of systems with fault-prone measurements. Another approach is based on treating the discretely-distributed fault indicators as nuisance parameters, known to the observer. Originally proposed in [30] for a general class of systems, this approach was applied in [31] to target tracking using sensors with detection probability smaller than one. Using the fact that the measurement interruption process is white the authors derived an approximation of the CRLB for the tracking errors. However, in the case of general (nonwhite) Markov sequences such derivation becomes cumbersome. Moreover, this lower bound cannot be evaluated in closed form, calling for the use of extensive Monte-Carlo simulations.

A new approach for the calculation of a CRLB-type lower bound for hybrid systems is presented in this paper. The approach is based on the approximation of the discrete distributions of the fault indicators by appropriate smooth distributions. Satisfying the regularity conditions, the approximating distributions allow the calculation of the CRLB using its sequential version for general discrete-time nonlinear systems [24]. The CRLB-type lower bound for the original hybrid system is then obtained via a limiting process applied to the approximating system. Unlike the result of [30], the bound presented herein can be evaluated in closed-form. Therefore, its application to complex systems entails only a modest computational load. Moreover, this result enables the practitioner to analytically evaluate the effects of various system parameters on the attainable estimation performance. The utility of the proposed lower bound, as well as the simplicity of its application, are demonstrated via a numerical example involving a GPS-aided navigation system.

The remainder of this paper is organized as follows. The system model is defined and the problem is formulated in Section II. The underlying idea behind the derivation of the proposed lower bound is presented and justified in Section III. The formal derivation procedure is then detailed in Section IV. The main result of this paper, namely, the lower bound for systems with fault-prone measurements, is presented and discussed in Section V. A numerical example demonstrating the application of the new lower bound to the assessment of navigation accuracy in a simple GPS-aided navigation system is presented in Section VI. Concluding remarks are offered in the final Section. To enhance readability, some auxiliary calculations and developments are deferred to Appendices.

II. Problem Formulation

Consider the system

$$x_{k+1} = \Phi_{k+1}x_k + \Psi_{k+1}u_{k+1} + G_{k+1}w_{k+1} \quad x_k \in \mathbb{R}^n \tag{1a}$$

$$y_{k+1} = H_{k+1}(\gamma_{k+1})x_{k+1} + v_{k+1} \qquad y_k \in \mathbb{R}^m \tag{1b}$$

where $\{u_k\}_{k=1}^{\infty}$ is a sequence of (known) deterministic inputs, $\{w_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ are white sequences of process and measurement noise, respectively, with $w_k \sim \mathcal{N}(0, Q_k), v_k \sim \mathcal{N}(0, R_k)$ and $R_k > 0, x_0 \sim \mathcal{N}(\hat{x}_0, \Sigma_0)$ is the random initial state with $\Sigma_0 > 0, \gamma_k = [\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(N)}]^T$ is the vector of mode variables, and every sequence $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$ is a Bernoulli Markov chain with distribution

$$\Pr\left\{\gamma_{k+1}^{(i)} = 1 \,|\, \gamma_k^{(i)} = j\right\} = P_{1j}^{(i)} \quad P_{1j}^{(i)} \neq 0, 1, \quad j \in \{0, 1\} \tag{2a}$$

$$\Pr\{\gamma_0^{(i)} = 1\} = \Pi_0^{(i)}. \tag{2b}$$

It is assumed that the chains, $\{\gamma_k^{(i)}\}_{k=0}^{\infty}$, are homogenous, i.e., the transition probabilities $P_{1j}^{(i)}$ are constant in time.

It is also assumed that $\{w_k\}_{k=1}^{\infty}, \{v_k\}_{k=1}^{\infty}, x_0$ and the chains $\{\gamma_k^{(i)}\}_{k=0}^{\infty}, i = 1, 2, \ldots, N$, are mutually independent. In addition, the observation matrix is assumed to satisfy the following structural constraint:

$$H_k(\gamma_k) = H_k^{(0)} + \sum_{i=1}^{N} H_k^{(i)}\gamma_k^{(i)}. \tag{3}$$

The hybrid system defined previously is denoted by $\mathcal{H}$.

For notational simplicity the explicit time-dependence is suppressed in the sequel in all places where it is clear by context. Also, the transition matrix Φ is assumed to be nonsingular.

The model defined above can represent a wide class of systems in the area of fault detection and isolation. In these systems, the state vector x_k comprises two parts: The primary part is associated with the system dynamics and the secondary part is associated with the dynamics of the sensors when sensor faults occur. These fault states can describe various kinds of sensors' faulty behavior, e.g., measurement biases, or additive faulty measurement noises (white or colored). The interruption variables $\gamma_k^{(i)}$ play the role of fault indicators. An example of such a system is presented in Section VI.

The following definitions will be used in the sequel. Let

$$z_k \triangleq \begin{bmatrix} x_k \\ \gamma_k \end{bmatrix} \tag{4}$$

be the augmented state vector of the system. Denote, also, the accumulated history of the measurements as

$$\mathcal{Y}_k \triangleq \left[y_1^T, y_2^T, \ldots, y_k^T\right]^T \tag{5}$$

Any estimator of a vector χ_k based on the measurement history $\mathcal{Y}_k$ is denoted as $\hat{\chi}_{k\,|\,k}$. Let

$$\Pi_k^{(i)} \triangleq \Pr\left\{\gamma_k^{(i)} = 1\right\} \tag{6}$$

and

$$\Pi_k \triangleq \left[\Pi_k^{(1)}, \Pi_k^{(2)}, \ldots, \Pi_k^{(N)}\right]. \tag{7}$$

In addition, define the operator Δ_ξ^η as

$$\Delta_\xi^\eta \triangleq \nabla_\xi \nabla_\eta^T \tag{8}$$

where

$$\nabla_\zeta \triangleq \left[\begin{matrix} \frac{\partial}{\partial \zeta_1} & \frac{\partial}{\partial \zeta_2} & \cdots & \frac{\partial}{\partial \zeta_n} \end{matrix}\right]^T \qquad \forall \zeta \in \mathbb{R}^n \tag{9}$$

is the gradient operator. Finally, for presentation clarity, the notational convention of [17] is adopted, according to which lower case and upper case letters are used to denote the pdf's associated random variables and their realizations, respectively.

The goal of this work is to derive a lower bound for the estimation error covariance matrix of the augmented state vector (4).

III. Underlying Idea

In this section, the idea underlying the derivation of the new lower bound is proposed and justified. The derivation is based on the sequential form of the CRLB [24], which is summarized in the next theorem.

Theorem 3.1 (Tichavský, Muravchik, and Nehorai, 1998): Let $\{z_k\}_{k=0}^\infty$ denote a Markovian state process. Let $\{y_k\}_{k=1}^\infty$ denote the corresponding measurement process, assumed to be of the form

$$y_k = h_k(z_k, v_k) \tag{10}$$

where $\{v_k\}_{k=1}^\infty$ is a white sequence independent of the process noise. Then

$$E[(z_k - \hat{z}_{k\,|\,k})(z_k - \hat{z}_{k\,|\,k})^T] \geqslant J_k^{-1} \tag{11}$$

where J_k, termed Fisher information submatrix in [24], is computed sequentially according to the recursion

$$J_{k+1} = D_k^{22} - D_k^{21}\left(J_k + D_k^{11}\right)^{-1} D_k^{12} \tag{12a}$$

$$J_0 = E\left[-\Delta_{z_0}^{z_0} \ln p_{z_0}(z_0)\right] \tag{12b}$$

with

$$D_k^{11} = E\left[-\Delta_{z_k}^{z_k} \ln p_{z_{k+1}\,|\,z_k}(z_{k+1}\,|\,z_k)\right] \tag{13a}$$

$$D_k^{12} = E\left[-\Delta_{z_k}^{z_{k+1}} \ln p_{z_{k+1}\,|\,z_k}(z_{k+1}\,|\,z_k)\right] = {D_k^{21}}^T \tag{13b}$$

$$D_k^{22} = E\left[-\Delta_{z_{k+1}}^{z_{k+1}} \ln p_{z_{k+1}\,|\,z_k}(z_{k+1}\,|\,z_k)\right] + E\left[-\Delta_{z_{k+1}}^{z_{k+1}} \ln p_{y_{k+1}\,|\,z_{k+1}}(y_{k+1}\,|\,z_{k+1})\right]. \tag{13c}$$

Remark 3.1: It follows from the derivation presented in [24] that the matrices D_k^{ij} are blocks of the Fisher information matrix corresponding to the joint distribution of the history of all the states $\{z_l\}_{l=0}^{k+1}$ and all the measurements $\mathcal{Y}_{k+1}$. Moreover, J_{k+1}^{-1} is the $(k+1, k+1)$ block of the inverse of this global Fisher information matrix.

Remark 3.2: If the CRLB regularity conditions are satisfied, the resulting Fisher information matrix is nonsingular [18]. It follows, therefore, that the Fisher information submatrix J_{k+1} is also nonsingular.

Straightforward use of the sequential CRLB, presented above, to compute an estimation bound for both the state x_k and the fault γ_k of the system $\mathcal{H}$, defined in Section II, is impossible. This follows from the discrete nature of the distribution of the fault vector, which violates the requirement that the joint pdf of the state vector and the measurement vector be twice differentiable. Moreover, since in the system addressed in this paper the measurements are related to the state through

$$y_k = h_k(x_k, \gamma_k, v_k) \tag{14}$$

the sequential CRLB cannot be applied to the primary state, x_k, alone, unless $\{\gamma_k\}_{k=0}^\infty$ is white, because the structural requirement (10) is not satisfied.

To circumvent the aforementioned problem, it is proposed herein to adopt the following two-stage derivation procedure. In the first stage, the original hybrid system $\mathcal{H}$ is approximated by a continuous system which is identical to $\mathcal{H}$ except that the discrete distribution of γ_k is replaced by a continuous one. Specifically, the Bernoulli Markov chain $\{\gamma_k^{(i)}\}_{k=0}^\infty$ is replaced by a continuously distributed Markov process $\{\tilde{\gamma}_k^{(i)}\}_{k=0}^\infty$. The transition and initial pdfs of this new process are defined as

$$p_{\tilde{\gamma}_{k+1}^{(i)}\,|\,\tilde{\gamma}_k^{(i)}}(\Gamma_{k+1}\,|\,\Gamma_k) = \frac{1}{\sqrt{2\pi}\sigma}\left[P^{(i)}(\Gamma_k) e^{-\frac{(\Gamma_{k+1}-1)^2}{2\sigma^2}} + \left(1 - P^{(i)}(\Gamma_k)\right) e^{-\frac{\Gamma_{k+1}^2}{2\sigma^2}}\right] \tag{15a}$$

$$p_{\tilde{\gamma}_0}(\Gamma_0) = \frac{1}{\sqrt{2\pi}\sigma}\left[\Pi_0^{(i)} e^{-\frac{(\Gamma_0-1)^2}{2\sigma^2}} + \left(1 - \Pi_0^{(i)}\right) e^{-\frac{\Gamma_0^2}{2\sigma^2}}\right] \tag{15b}$$

where σ is some positive real parameter, $\Pi_0^{(i)}$ is the probability defined in (2b) and the function $P^{(i)}(\Gamma)$ has the following properties:

1) $P^{(i)}(\Gamma)$ is twice continuously differentiable;
2) the first two derivatives of $P^{(i)}(\Gamma)$, namely, $P^{(i)}(\Gamma)'$ and $P^{(i)}(\Gamma)''$, are bounded;
3) $P^{(i)}(\Gamma) = P_{11}^{(i)}$ and $P^{(i)}(\Gamma)' = 0\ \forall \Gamma \geqslant 1$;
4) $P^{(i)}(\Gamma) = P_{10}^{(i)}$ and $P^{(i)}(\Gamma)' = 0\ \forall \Gamma \leqslant 0$;
5) $P^{(i)}(\Gamma)$ is monotonic.

Notice that each pdf defined in (15) has the form of a two-peak function. The steepness of the peaks is determined by the parameter σ. Intuitively it is clear that the smaller the value of σ, the better the approximation of the discrete distribution (2) by the continuous distribution (15). A thorough discussion of the properties of the distribution (15) can be found in [32].

Now, denote by $\mathcal{C}(\sigma)$ the continuous system resulting from replacing in $\mathcal{H}$ the discrete distribution (2) by the continuous distribution (15) for some $\sigma > 0$. As shown in Appendix B, the resulting joint pdf of the measurements and the estimated variables satisfies the regularity conditions even in their most restrictive form [17, p. 72]. Thus, the recursive CRLB corresponding to $\mathcal{C}(\sigma)$ can be computed for any $\sigma > 0$. In the second stage of the derivation, a limiting process is applied to this CRLB as $\sigma \rightarrow 0^+$. As the next two theorems show, this two-stage procedure leads to a lower bound for the original system $\mathcal{H}$, since: 1) the continuous system $\mathcal{C}(\sigma)$ approximates the hybrid system $\mathcal{H}$ in the sense that the estimation error covariance matrices resulting from the application of any admissible estimator to both systems can be made arbitrarily close by letting $\sigma \rightarrow 0^+$, and 2) the CRLB obtained for $\mathcal{C}(\sigma)$ provides a lower bound for the original hybrid system $\mathcal{H}$ via a limiting procedure, as $\sigma \rightarrow 0^+$.

To present the next theorems let $\mathcal{E}$ be any estimator satisfying the CRLB requirements. Let $S_{\mathcal{H}}(k)$ and $S_{\mathcal{C}(\sigma)}(k)$ be the estimation error covariance matrices resulting from the application of $\mathcal{E}$ to $\mathcal{H}$ and to $\mathcal{C}(\sigma)$ using measurements up to and including time k, respectively, where $k < \infty$. In addition, let $J_k(\sigma)$ be the Fisher information submatrix obtained by applying the sequential CRLB to $\mathcal{C}(\sigma)$.

Theorem 3.2: The system $\mathcal{H}$ can be approximated by systems with continuously distributed interruption variables [as in (15)] to an arbitrary degree of accuracy, in the sense that

$$\lim_{\sigma \rightarrow 0^+} S_{\mathcal{C}(\sigma)}(k) = S_{\mathcal{H}}(k). \tag{16}$$

Proof: The proof makes use of some results that are presented in Appendix A (and numbered accordingly). Let

$$e_k \triangleq [\,(x_k - \hat{x}_{k\,|\,k})^T \quad (\gamma_k - \hat{\gamma}_{k\,|\,k})^T\,]^T \tag{17}$$

$$\tilde{e}_k \triangleq [\,(x_k - \hat{x}_{k\,|\,k})^T \quad (\tilde{\gamma}_k - \hat{\gamma}_{k\,|\,k})^T\,]^T. \tag{18}$$

Then, the smoothing property of conditional expectation yields

$$S_{\mathcal{H}}(k) = E\left[e_k e_k^T\right] = E\left[E\left[e_k e_k^T \,|\, \gamma_1, \ldots, \gamma_k\right]\right] \tag{19a}$$

$$S_{\mathcal{C}(\sigma)}(k) = E\left[\tilde{e}_k \tilde{e}_k^T\right] = E\left[E\left[\tilde{e}_k \tilde{e}_k^T \,|\, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_k\right]\right] \tag{19b}$$

where the inner conditional expectations in both (19a) and (19b) are identical matrix functions of some sets of values of γ_i's and $\tilde{\gamma}_i$'s, respectively. It will be shown now that the (i,j) element of these expectations, denoted by A_{ij}, is a continuous function of the conditioning variables.

First, it can be seen that every element of the conditional expectations can be expressed as a second-order polynomial of the conditioning variables with coefficients of the form

$$\int_{-\infty}^{+\infty} g(X_k, \Upsilon_k) \times p_{x_k, \mathcal{Y}_k \,|\, \gamma_1, \ldots, \gamma_k}(X_k, \Upsilon_k \,|\, \Gamma_1, \ldots, \Gamma_k) \mathrm{d}X_k \mathrm{d}\Upsilon_k \tag{20}$$

where the function $g(X_k, \Upsilon_k)$ is determined by the particular estimator. Therefore, it is sufficient to show that the integrals defined by (20) are continuous functions of Γ_i's. To this end it is assumed that

$$\left| \int_{-\infty}^{+\infty} g(X_k, \Upsilon_k) \times p_{x_k, \mathcal{Y}_k \,|\, \gamma_1, \ldots, \gamma_k}(X_k, \Upsilon_k \,|\, \Gamma_1, \ldots, \Gamma_k) \mathrm{d}X_k \mathrm{d}\Upsilon_k \right| < \infty \tag{21a}$$

$$\int_{-\infty}^{+\infty} g(X_k, \Upsilon_k)^2 \times p_{x_k, \mathcal{Y}_k \,|\, \gamma_1, \ldots, \gamma_k}(X_k, \Upsilon_k \,|\, \Gamma_1, \ldots, \Gamma_k) \mathrm{d}X_k \mathrm{d}\Upsilon_k < \infty \tag{21b}$$

meaning that the estimation errors have finite second- and fourth-order moments. Now, the conditional pdf $p_{x_k, \mathcal{Y}_k \,|\, \gamma_1, \ldots, \gamma_k}(X_k, \Upsilon_k \,|\, \Gamma_1, \ldots, \Gamma_k)$ is Gaussian. Since γ_i's appear linearly in (3), the mean and the covariance matrix of this distribution are continuous functions of γ_i's. On the other hand, by Lemma A.1, the integral defined in (20) and satisfying conditions (21) is a continuous function of the mean and the covariance matrix. Therefore, this integral is a continuous function of γ_i's.

To demonstrate (16), it is sufficient to show that

$$\lim_{\sigma \rightarrow 0^+} \int_{-\infty}^{+\infty} A_{ij} p_{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k}(\Gamma_1, \ldots, \Gamma_k) \mathrm{d}\Gamma_1 \cdots \mathrm{d}\Gamma_k = \sum_{\Gamma_1, \ldots, \Gamma_k} A_{ij} \Pr\{\gamma_1 = \Gamma_1, \ldots, \gamma_k = \Gamma_k\}. \tag{22}$$

First, notice that due to the Markov property

$$\begin{aligned} p_{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k}(\Gamma_1, \ldots, \Gamma_k) &= p_{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{k-1}}(\Gamma_1, \ldots, \Gamma_{k-1}) p_{\tilde{\gamma}_k \,|\, \tilde{\gamma}_{k-1}}(\Gamma_k \,|\, \Gamma_{k-1}) \\ &= p_{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{k-1}}(\Gamma_1, \ldots, \Gamma_{k-1}) \prod_{l=1}^{N} p_{\tilde{\gamma}_k^{(l)} \,|\, \tilde{\gamma}_{k-1}^{(l)}} \left(\Gamma_k^{(l)} \,|\, \Gamma_{k-1}^{(l)}\right) \end{aligned} \tag{23}$$

where each of the conditional pdfs is given by (15a). It will be shown now that

$$\lim_{\sigma \rightarrow 0^+} \int_{-\infty}^{+\infty} A_{ij} p_{\tilde{\gamma}_k^{(l)} \,|\, \tilde{\gamma}_{k-1}^{(l)}} \left(\Gamma_k^{(l)} \,|\, \Gamma_{k-1}^{(l)}\right) \mathrm{d}\Gamma_k^{(l)} = \sum_{\Gamma_k^{(l)}=0}^{1} A_{ij} \Pr\left\{\gamma_k^{(l)} = \Gamma_k^{(l)} \,|\, \gamma_{k-1}^{(l)} = \Gamma_{k-1}^{(l)}\right\}. \tag{24}$$

Notice that

$$\begin{aligned} \int_{-\infty}^{+\infty} & A_{ij} p_{\tilde{\gamma}_k^{(l)} \,|\, \tilde{\gamma}_{k-1}^{(l)}} \left(\Gamma_k^{(l)} \,|\, \Gamma_{k-1}^{(l)}\right) \mathrm{d}\Gamma_k^{(l)} \\ &= \frac{1}{\sqrt{2\pi}\sigma} \left[P^{(l)}\left(\Gamma_{k-1}^{(l)}\right) \int_{-\infty}^{+\infty} A_{ij} e^{-\frac{\left(\Gamma_k^{(l)}-1\right)^2}{2\sigma^2}} \mathrm{d}\Gamma_k^{(l)} \right. \\ &\quad \left. + \left(1 - P^{(l)}\left(\Gamma_{k-1}^{(l)}\right)\right) \int_{-\infty}^{+\infty} A_{ij} e^{-\frac{\Gamma_k^{(l)2}}{2\sigma^2}} \mathrm{d}\Gamma_k^{(l)} \right]. \end{aligned} \tag{25}$$

As previously mentioned, A_{ij} is a continuous function of $\Gamma_k^{(l)}$. In addition, by Lemma A.2, being an element of a positive–semidefinite matrix, A_{ij} is absolutely integrable with respect to the conditional distribution of $\tilde{\gamma}_k^{(l)} \mid \tilde{\gamma}_{k-1}^{(l)}$. Therefore, by Lemma A.3

$$\lim_{\sigma \to 0^+} \int_{-\infty}^{+\infty} A_{ij} p_{\tilde{\gamma}_k^{(l)} \mid \tilde{\gamma}_{k-1}^{(l)}} \left(\Gamma_k^{(l)} \,\middle|\, \Gamma_{k-1}^{(l)} \right) \mathrm{d}\Gamma_k^{(l)} = P^{(l)}\left(\Gamma_{k-1}^{(l)}\right) A_{ij}|_{\Gamma_k^{(l)}=1} + \left(1 - P^{(l)}\left(\Gamma_{k-1}^{(l)}\right)\right) \times A_{ij}|_{\Gamma_k^{(l)}=0} \tag{26}$$

and (24) is obtained recalling that

$$P^{(l)}(0) = P_{10}^{(l)} \quad P^{(l)}(1) = P_{11}^{(l)}. \tag{27}$$

Applying these arguments for each $\gamma_\xi^{(l)}$ and using the fact that $P^{(l)}(\Gamma)$ are continuous functions of Γ finally yields (22). ■

Theorem 3.3: The Fisher information submatrix $J_k(\sigma)$, obtained by applying the CRLB to the approximating continuous system $\mathcal{C}(\sigma)$, provides an estimation error lower bound for the hybrid system $\mathcal{H}$ via a limiting process, namely

$$S_{\mathcal{H}}(k) \geqslant \lim_{\sigma \to 0^+} J_k(\sigma)^{-1}. \tag{28}$$

Proof: Since $J_k(\sigma)$ is, by definition, the Fisher information submatrix computed for the approximating continuously distributed system for some $\sigma > 0$, then

$$S_{\mathcal{C}(\sigma)}(k) \geqslant J_k(\sigma)^{-1}. \tag{29}$$

Therefore, Theorem 3.2 yields

$$S_{\mathcal{H}}(k) = \lim_{\sigma \to 0^+} S_{\mathcal{C}(\sigma)}(k) \geqslant \lim_{\sigma \to 0^+} J_k(\sigma)^{-1}. \tag{30}$$

■

IV. Lower Bound Derivation

This section presents a derivation of the new lower bound, based on the underlying idea proposed in Section III. The new lower bound itself, which constitutes the paper's main result, is presented in the next section. The derivation comprises two steps: 1) application of the sequential CRLB to the approximating continuous system $\mathcal{C}(\sigma)$ for some $\sigma > 0$, and 2) computation of the CRLB limit as $\sigma \to 0^+$. For the sake of brevity only the main highlights of the derivation are presented. The interested reader is referred to [32] for a detailed derivation.

To facilitate the derivation the matrix GQG^T is initially assumed to be nonsingular. This assumption is relaxed in the sequel via a procedure proposed in [24].

A. CRLB for $\sigma > 0$

Analogously to (4), let

$$\tilde{z}_k \triangleq \begin{bmatrix} x_k \\ \tilde{\gamma}_k \end{bmatrix}. \tag{31}$$

Since x_k and the elements of $\tilde{\gamma}_k$ are mutually independent, the augmented state vector $\tilde{z}_k$ has the marginal density

$$p_{\tilde{z}_k}(Z_k) = p_{x_k,\tilde{\gamma}_k}(X_k, \Gamma_k) = p_{x_k}(X_k) \prod_{i=1}^{N} p_{\tilde{\gamma}_k^{(i)}}\left(\Gamma_k^{(i)}\right) \tag{32}$$

and the following transitional density:

$$p_{\tilde{z}_{k+1} \mid \tilde{z}_k}(Z_{k+1} \mid Z_k) = p_{x_{k+1} \mid x_k}(X_{k+1} \mid X_k) \prod_{i=1}^{N} p_{\tilde{\gamma}_{k+1}^{(i)} \mid \tilde{\gamma}_k^{(i)}}\left(\Gamma_{k+1}^{(i)} \mid \Gamma_k^{(i)}\right). \tag{33}$$

The distribution of x_k is Gaussian with

$$x_{k+1} \mid x_k \sim \mathcal{N}(\Phi x_k + \Psi u_{k+1}, GQG^T). \tag{34}$$

Therefore

$$\ln p_{\tilde{z}_{k+1} \mid \tilde{z}_k}(Z_{k+1} \mid Z_k) = l_{xx}(X_{k+1} \mid X_k) + \sum_{i=1}^{N} l_{\gamma\gamma}^{(i)}\left(\Gamma_{k+1}^{(i)} \mid \Gamma_k^{(i)}\right) \tag{35}$$

where

$$l_{xx}(X_{k+1} \mid X_k) \triangleq \ln p_{x_{k+1} \mid x_k}(X_{k+1} \mid X_k) = \ln \frac{1}{\sqrt{(2\pi)^n \det(GQG^T)}} - \frac{1}{2}[X_{k+1} - \Phi X_k - \Psi u_{k+1}]^T \left(GQG^T\right)^{-1} \times [X_{k+1} - \Phi X_k - \Psi u_{k+1}] \tag{36}$$

and

$$l_{\gamma\gamma}^{(i)}\left(\Gamma_{k+1}^{(i)} \mid \Gamma_k^{(i)}\right) \triangleq \ln p_{\tilde{\gamma}_{k+1}^{(i)} \mid \tilde{\gamma}_k^{(i)}}\left(\Gamma_{k+1}^{(i)} \mid \Gamma_k^{(i)}\right). \tag{37}$$

The distribution of the measurements conditioned on $\tilde{z}_k$ is also Gaussian, hence

$$\ln p_{y_{k+1} \mid \tilde{z}_{k+1}}(Y_{k+1} \mid Z_{k+1}) = \ln \frac{1}{\sqrt{(2\pi)^m \det R}} - \frac{1}{2}[Y_{k+1} - H(\Gamma_{k+1})X_{k+1}]^T R^{-1} \times [Y_{k+1} - H(\Gamma_{k+1})X_{k+1}]. \tag{38}$$

Using (35) and (38) in (13), the following expressions for the D_k^{ij} matrices, defined in Theorem 3.1, are obtained

$$D_k^{11} = \begin{bmatrix} \Phi^T(GQG^T)^{-1}\Phi & 0 \\ 0 & D_{\gamma\gamma}^{11}(k) \end{bmatrix} \tag{39a}$$

$$D_k^{12} = -\begin{bmatrix} \Phi^T(GQG^T)^{-1} & 0 \\ 0 & D_{\gamma\gamma}^{12}(k) \end{bmatrix} \tag{39b}$$

$$D_k^{22} = \begin{bmatrix} D_{xx}^{22}(k) + (GQG^T)^{-1} & D_{x\gamma}^{22}(k) \\ D_{x\gamma}^{22}(k)^T & \frac{1}{\sigma^2}I + U(k) + V(k) \end{bmatrix} \tag{39c}$$

In (39), the following terms are used.

1) $D_{\gamma\gamma}^{11}(k)$ is a diagonal matrix whose ith diagonal entry is

$$D_{\gamma\gamma}^{11}(k)_{ii} \triangleq E\left[-\frac{\partial^2 l_{\gamma\gamma}\left(\tilde{\gamma}_{k+1}^{(i)} \,|\, \tilde{\gamma}_k^{(i)}\right)}{\partial \tilde{\gamma}_k^{(i)^2}}\right]. \tag{40}$$

2) $D_{\gamma\gamma}^{12}(k)$ is a diagonal matrix whose ith diagonal entry is

$$D_{\gamma\gamma}^{12}(k)_{ii} \triangleq E\left[\frac{\partial^2 l_{\gamma\gamma}\left(\tilde{\gamma}_{k+1}^{(i)} \,|\, \tilde{\gamma}_k^{(i)}\right)}{\partial \tilde{\gamma}_{k+1}^{(i)} \partial \tilde{\gamma}_k^{(i)}}\right]. \tag{41}$$

3) The matrix $D_{xx}^{22}(k)$ is given by

$$D_{xx}^{22}(k) \triangleq E\left[H(\tilde{\gamma}_{k+1})^T R^{-1} H(\tilde{\gamma}_{k+1})\right]. \tag{42}$$

4) The (i,j) element of $V(k)$ is

$$V(k)_{ij} \triangleq E\left[x_{k+1}^T H^{(i)^T} R^{-1} H^{(j)} x_{k+1}\right]. \tag{43}$$

5) The jth column of $D_{x\gamma}^{22}(k)$ is

$$D_{x\gamma}^{22}(k)_j = E\left[H(\tilde{\gamma}_{k+1})\right]^T R^{-1} H^{(j)} E\left[x_{k+1}\right]. \tag{44}$$

6) $U(k)$ is a diagonal matrix whose ith diagonal entry is

$$U(k)_{ii} = E\left[-\frac{\partial^2 l_{\gamma\gamma}\left(\tilde{\gamma}_{k+1}^{(i)} \,\middle|\, \tilde{\gamma}_k^{(i)}\right)}{\partial \tilde{\gamma}_{k+1}^{(i)^2}}\right] - \frac{1}{\sigma^2}. \tag{45}$$

It is shown in [32] that

$$\lim_{\sigma \to 0^+} D_{\gamma\gamma}^{11}(k) = 0 \tag{46a}$$
$$\lim_{\sigma \to 0^+} D_{\gamma\gamma}^{12}(k) = 0 \tag{46b}$$
$$\lim_{\sigma \to 0^+} U(k) = 0. \tag{46c}$$

Now, substituting (39) into (12a) yields the following propagation formula for the Fisher information submatrix:

$$\begin{aligned} J_{k+1}(\sigma) = &\begin{bmatrix} D_{xx}^{22}(k) + (GQG^T)^{-1} & D_{x\gamma}^{22}(k) \\ D_{x\gamma}^{22}(k)^T & \frac{1}{\sigma^2} I + U(k) + V(k) \end{bmatrix} \\ &- \begin{bmatrix} (GQG^T)^{-1}\Phi & 0 \\ 0 & D_{\gamma\gamma}^{12}(k)^T \end{bmatrix} \\ &\times \left(J_k(\sigma) + \begin{bmatrix} \Phi^T (GQG^T)^{-1}\Phi & 0 \\ 0 & D_{\gamma\gamma}^{11}(k) \end{bmatrix}\right)^{-1} \\ &\times \begin{bmatrix} \Phi^T (GQG^T)^{-1} & 0 \\ 0 & D_{\gamma\gamma}^{12}(k) \end{bmatrix}. \end{aligned} \tag{47}$$

Since in the approximating system $\mathcal{C}(\sigma)$ the CRLB regularity conditions are satisfied, Remark 3.2 renders the global Fisher information matrix nonsingular. Therefore, the Fisher information submatrix $J_k(\sigma)$ is also nonsingular.

To avoid the need to explicitly invert GQG^T in (47), the expression (47) can be rewritten in the following form, using the well-known matrix inversion lemma [33, p. 19]:

$$\begin{aligned} &J_{k+1}(\sigma) \\ &= \begin{bmatrix} D_{xx}^{22}(k) & D_{x\gamma}^{22}(k) \\ D_{x\gamma}^{22}(k)^T & \left(\frac{1}{\sigma^2} - 1\right) I + U(k) + V(k) \end{bmatrix} \\ &+ \left(\begin{bmatrix} GQG^T & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & D_{\gamma\gamma}^{12}(k)^T \end{bmatrix}\right. \\ &\times \left\{J_k(\sigma) + \begin{bmatrix} 0 & 0 \\ 0 & D_{\gamma\gamma}^{11}(k) - D_{\gamma\gamma}^{12}(k) D_{\gamma\gamma}^{12}(k)^T \end{bmatrix}\right\}^{-1} \\ &\times \left.\begin{bmatrix} \Phi^T & 0 \\ 0 & D_{\gamma\gamma}^{12}(k) \end{bmatrix}\right)^{-1}. \end{aligned} \tag{48}$$

B. Limiting Case

The limiting value of the CRLB for $\sigma \to 0^+$ is presented in the following theorem.

Theorem 4.1: For every finite time instant $k \geq 0$, the inverse of the limiting Fisher information submatrix satisfies the following relation:

$$\lim_{\sigma \to 0^+} J_k(\sigma)^{-1} = \begin{bmatrix} J_{xx}^{\star}(k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{49}$$

where $J_{xx}^{\star}(k) \in \mathbb{R}^{n \times n}$ is a positive–definite matrix satisfying the following recursion:

$$\begin{aligned} J_{xx}^{\star}(k+1) = &\, E\left[H(\gamma_{k+1})^T R^{-1} H(\gamma_{k+1})\right] \\ &+ \left(\Phi J_{xx}^{\star}(k)^{-1} \Phi^T + GQG^T\right)^{-1} \end{aligned} \tag{50a}$$
$$J_{xx}^{\star}(0) = \Sigma_0^{-1}. \tag{50b}$$

In (50a) γ_{k+1} is the discretely-distributed fault indicator vector of the original hybrid system.

Proof: The proof uses mathematical induction. First, using (12b) and following the derivation procedure of (39c), it can be shown that, for $k = 0$

$$J_0(\sigma) = \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & \frac{1}{\sigma^2} I + U^{\star} \end{bmatrix}. \tag{51}$$

In (51) $U^{\star}$ is a diagonal matrix, whose ith diagonal entry is

$$U_{ii}^{\star} = E\left[-\frac{\partial^2 \ln p_{\tilde{\gamma}_0^{(i)}}\left(\tilde{\gamma}_0^{(i)}\right)}{\partial \tilde{\gamma}_{k+1}^{(i)^2}}\right] - \frac{1}{\sigma^2} \tag{52}$$

that satisfies

$$\lim_{\sigma \to 0^+} U^{\star} = 0. \tag{53}$$

Therefore

$$\lim_{\sigma \to 0^+} J_0(\sigma)^{-1} = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} \tag{54}$$

which implies both (49), for $k=0$, and (50b). Notice that the positive–definiteness of $J^{\star}_{xx}(0)$ follows from that of Σ_0.

Now, assuming that the Theorem is valid for the finite time instant k, it will be shown that it also holds for time instant $k+1$. The induction recursion is proved in the following three steps.

Step 1) Using continuity arguments, the induction assumption, the matrix inversion lemma and the fact that, by (46a) and (46b)

$$\lim_{\sigma\to 0^+}\left[D^{11}_{\gamma\gamma}(k)-D^{12}_{\gamma\gamma}(k)D^{12}_{\gamma\gamma}(k)^T\right]=0 \tag{55}$$

yields

$$\lim_{\sigma\to 0^+}\left\{J_k(\sigma)+\begin{bmatrix}0 & 0\\ 0 & D^{11}_{\gamma\gamma}(k)-D^{12}_{\gamma\gamma}(k)D^{12}_{\gamma\gamma}(k)^T\end{bmatrix}\right\}^{-1} = \begin{bmatrix}J^{\star}_{xx}(k)^{-1} & 0\\ 0 & 0\end{bmatrix}. \tag{56}$$

Step 2) Applying the matrix inversion lemma to the inverse on the right-hand side (RHS) of (48) and using (46b) and (56) yields

$$\lim_{\sigma\to 0^+}\left(\begin{bmatrix}GQG^T & 0\\ 0 & I\end{bmatrix}+\begin{bmatrix}\Phi & 0\\ 0 & D^{12}_{\gamma\gamma}(k)^T\end{bmatrix}\times\left\{J_k(\sigma)+\begin{bmatrix}0 & 0\\ 0 & D^{11}_{\gamma\gamma}(k)-D^{12}_{\gamma\gamma}(k)D^{12}_{\gamma\gamma}(k)^T\end{bmatrix}\right\}^{-1}\times\begin{bmatrix}\Phi^T & 0\\ 0 & D^{12}_{\gamma\gamma}(k)\end{bmatrix}\right)^{-1} = \begin{bmatrix}\left(\Phi J^{\star}_{xx}(k)^{-1}\Phi^T+GQG^T\right)^{-1} & 0\\ 0 & I\end{bmatrix}. \tag{57}$$

Step 3) Finally, it is shown that

$$\lim_{\sigma\to 0^+} J^{-1}_{k+1}(\sigma)=\begin{bmatrix}J^{\star}_{xx}(k+1)^{-1} & 0\\ 0 & 0\end{bmatrix} \tag{58}$$

where $J^{\star}_{xx}(k+1)$ is calculated using (50a). Notice that all the terms on the RHS of (48), except for $(1/\sigma^2)I$, are finite for $k<\infty$. Therefore, using (48)

$$\lim_{\sigma\to 0^+} J^{-1}_{k+1}(\sigma)=\lim_{\sigma\to 0^+}\begin{bmatrix}\Theta & \Lambda\\ \Lambda^T & \frac{1}{\sigma^2}I+\Omega\end{bmatrix}^{-1} \tag{59}$$

with implied definitions of the finite matrices Θ, Λ, and Ω. Since

$$\lim_{\sigma\to 0^+}\Theta=\lim_{\sigma\to 0^+}E\left[H(\tilde\gamma_{k+1})^TR^{-1}H(\tilde\gamma_{k+1})\right] + \left(\Phi J^{\star}_{xx}(k)^{-1}\Phi^T+GQG^T\right)^{-1} \tag{60}$$

the matrix Θ is nonsingular for sufficiently small values of σ. Similarly, for sufficiently small values of σ the term $1/\sigma^2 I+\Omega$ is also nonsingular. Using Schur's formula for the inversion of partitioned matrices [33, p. 18] yields

$$\lim_{\sigma\to 0^+}\begin{bmatrix}\Theta & \Lambda\\ \Lambda^T & \frac{1}{\sigma^2}I+\Omega\end{bmatrix}^{-1}=\begin{bmatrix}(\lim_{\sigma\to 0^+}\Theta)^{-1} & 0\\ 0 & 0\end{bmatrix}. \tag{61}$$

In addition, the expression $E[H(\tilde\gamma_{k+1})^TR^{-1}H(\tilde\gamma_{k+1})]$ depends only on the first and second moments of $\tilde\gamma_{k+1}$. Therefore

$$\lim_{\sigma\to 0^+}E\left[H(\tilde\gamma_{k+1})^TR^{-1}H(\tilde\gamma_{k+1})\right] = E\left[H(\gamma_{k+1})^TR^{-1}H(\gamma_{k+1})\right] \tag{62}$$

which yields

$$\lim_{\sigma\to 0^+}\Theta=E\left[H(\gamma_{k+1})^TR^{-1}H(\gamma_{k+1})\right] + \left(\Phi J^{\star}_{xx}(k)^{-1}\Phi^T+GQG^T\right)^{-1}. \tag{63}$$

Substituting (63) into (61) gives (58). The positive–definiteness of $J^{\star}_{xx}(k)$ follows from (50a) where the second term on the RHS is nonsingular. ■

Finally, the assumption regarding the regularity of GQG^T, made at the beginning of the derivation, is relaxed in the following theorem.

Theorem 4.2: Theorem 4.1 applies also in systems where GQG^T is singular.

Proof: The proof is motivated by a procedure shown in [24]. Replace GQG^T by $GQG^T+\varepsilon I$ for some small positive ε. This new matrix is nonsingular and, therefore, Theorem 4.1 applies. Sending ε to zero and using the continuity of the second term on the RHS of (50a) yield a CRLB-type lower bound for a singular GQG^T, which is also given by (50a). ■

V. Main Result

The main result of this paper is now stated in the following theorem.

Theorem 5.1: A CRLB-type lower bound for the system defined in Section II is given by

$$E\left[(x_k-\hat x_{k\,|\,k})(x_k-\hat x_{k\,|\,k})^T\right]\geqslant J^{\star}_{xx}(k)^{-1} \tag{64a}$$
$$E\left[(\gamma_k-\hat\gamma_{k\,|\,k})(\gamma_k-\hat\gamma_{k\,|\,k})^T\right]\geqslant 0 \tag{64b}$$

where $J^{\star}_{xx}(k)$ is computed using the recursion (50).

Proof: The Theorem follows immediately from Theorems 3.3 and 4.2. ■

A. Lower Bound Computational Algorithm

The lower bound presented in Theorem 5.1 can be easily evaluated using the following recursive algorithm.

1) Initialize the matrix $J^{\star}_{xx}(0)$ using (50b).
2) At each time step $k=0,1,2,\ldots$ update the *a priori* probabilities $\Pi^{(i)}_k, i=1,2,\ldots,N$

$$\Pi^{(i)}_{k+1}=P^{(i)}_{11}\Pi^{(i)}_k+P^{(i)}_{10}\left(1-\Pi^{(i)}_k\right). \tag{65}$$

3) Update the value of $J^{\star}_{xx}(k+1)$ using (50a), where the term $E[H(\gamma_{k+1})^T R^{-1} H(\gamma_{k+1})]$ can be computed as

$$
\begin{aligned}
E\left[H(\gamma_{k+1})^T R^{-1} H(\gamma_{k+1})\right] &= H_{k+1}(\Pi_{k+1})^T R_{k+1}^{-1} H_{k+1}(\Pi_{k+1}) \\
&\quad + \sum_{i=1}^{N} \Pi_{k+1}^{(i)} \left(1 - \Pi_{k+1}^{(i)}\right) H_{k+1}^{(i)}{}^{T} R_{k+1}^{-1} H_{k+1}^{(i)}. \quad (66)
\end{aligned}
$$

4) Finally, the CRLB-type lower bound is given by (64).

B. Lower Bound Properties

The result given by (64a) and (50) presents a relatively simple, nontrivial lower bound on the covariance of the estimation error of the state vector x_k. Similarly to the covariance recursion in the information form of the Kalman filter (KF), this bound can be easily computed and used to evaluate the performance of estimators designed for systems with fault-prone measurements.

On the other hand, the lower bound on the estimation error covariance of the fault indicators, given by (64b), is trivial. An intuitive reason for this result lies in the fact that the fault indicators $\gamma_k^{(i)}$, being discretely distributed parameters, are characterized by sharp changes in the cumulative distribution function. It has been reported previously in the literature [34] that the CRLB tends to zero in systems with sharp intensity functions. A direct conclusion from this observation is that in order to obtain a nontrivial lower bound on the discretely distributed fault indicators one must examine lower bounds that are not based on the CRLB. An example of such a lower bound has been recently presented in [35].

C. Comparison to a Monte-Carlo Lower Bound

Yet another lower bound can be obtained for the system by treating the fault vector sequence $\{\gamma_k\}_{k=0}^{\infty}$ as nuisance parameters that are known to the observer. Originally proposed in [30], this approach is as follows. Assume that the fault vector sequence is known. Under this assumption, the addressed system takes the form of an ordinary linear Gaussian system, yielding the following CRLB:

$$E\left[(x_k - \hat{x}_{k\,|\,k})(x_k - \hat{x}_{k\,|\,k})^T \,|\, \{\gamma_n\}_{n=1}^{k}\right] \geqslant J_{\mathrm{KF}}(k)^{-1} \quad (67)$$

where the Fisher information matrix is governed by the following recursion:

$$
\begin{aligned}
J_{\mathrm{KF}}(k+1) &= H(\gamma_{k+1})^T R^{-1} H(\gamma_{k+1}) \\
&\quad + \left(\Phi J_{\mathrm{KF}}(k)^{-1} \Phi^T + GQG^T\right)^{-1}. \quad (68)
\end{aligned}
$$

Taking mathematical expectation of both sides of (67) gives

$$E\left[(x_k - \hat{x}_{k\,|\,k})(x_k - \hat{x}_{k\,|\,k})^T\right] \geqslant E\left[J_{\mathrm{KF}}(k)^{-1}\right]. \quad (69)$$

The main drawback of this lower bound is the fact that it cannot be evaluated in closed-form. In practice, extensive Monte-Carlo simulations must be performed, rendering the computation of the bound rather difficult, compared to the result of Theorem 5.1. This bound is regarded in the present context as a Monte-Carlo lower bound (MCLB).

Remark 5.1: Since the MCLB explicitly assumes that the fault vector sequence is known to the observer, it does not take into account the effect of the fault vector estimation error on the state vector estimation accuracy.

The next proposition relates the lower bound presented in Theorem 5.1 to the MCLB.

Proposition 5.1: The lower bound presented in Theorem 5.1 is less tight than the MCLB, i.e.,

$$J^{\star}_{xx}(k)^{-1} \leqslant E\left[J_{\mathrm{KF}}(k)^{-1}\right]. \quad (70)$$

Proof: It is first proved that

$$E[J_{\mathrm{KF}}(k)] \leqslant J^{\star}_{xx}(k). \quad (71)$$

The proof is by mathematical induction. First, according to (50b)

$$J^{\star}_{xx}(0) = E[J_{\mathrm{KF}}(0)] = \Sigma_0^{-1}. \quad (72)$$

Assume now that inequality (71) holds for some k. Then, (50a), (68), and Lemma A.4 of Appendix A yield

$$
\begin{aligned}
E[J_{\mathrm{KF}}(k+1)] &\leqslant E\left[H(\gamma_{k+1})^T R^{-1} H(\gamma_{k+1})\right] \\
&\quad + (\Phi E[J_{\mathrm{KF}}(k)]^{-1} \Phi^T + GQG^T)^{-1} \leqslant J^{\star}_{xx}(k+1) \quad (73)
\end{aligned}
$$

so that (71) is established.

Now, it follows from (71) that

$$J^{\star}_{xx}(k)^{-1} \leqslant E[J_{\mathrm{KF}}(k)]^{-1}. \quad (74)$$

Finally, Lemma A.4 yields

$$J^{\star}_{xx}(k)^{-1} \leqslant E[J_{\mathrm{KF}}(k)]^{-1} \leqslant E\left[J_{\mathrm{KF}}(k)^{-1}\right]. \quad (75)$$

■

Remark 5.2: The arguments presented here can be interpreted as an alternative derivation of the lower bound given by Theorem 5.1. Moreover, this derivation is valid for a larger class of systems than that of Theorem 5.1 because no specific assumptions were made on the structure of the observation matrix $H(\gamma_k)$, as was done in (3). However, this derivation does not demonstrate the fact that the proposed lower bound is a limiting case of the CRLB.

Corollary 5.1: The lower bound presented in Theorem 5.1 does not take into account the effect of the fault vector estimation errors on the state vector estimation accuracy.

Proof: The Corollary follows from Proposition 5.1 and Remark 5.1. ■

VI. Numerical Example

A numerical proof-of-principle example is presented to demonstrate the applicability and utility of the new lower bound. The example involves a system comprising an inertial navigation system (INS), a GPS receiver and an additional position source, e.g., a terrain following system. Let $\delta p(t), \delta v(t)$, and $\delta\tau(t)$ denote the INS position error, velocity error and

GPS receiver clock drift, respectively. The following simplified dynamics is assumed:

$$\dot{\delta p}(t) = \delta v(t) \quad \delta p(t) \in \mathbb{R}^3 \quad \delta p(0) \sim \mathcal{N}\left(0, \Sigma_p^2 I\right) \tag{76a}$$

$$\dot{\delta v}(t) = n_v(t) \quad \delta v(t) \in \mathbb{R}^3 \quad \delta v(0) \sim \mathcal{N}\left(0, \Sigma_v^2 I\right) \tag{76b}$$

$$c\dot{\delta\tau}(t) = n_\tau(t) \quad c\delta\tau(0) \sim \mathcal{N}(0, \Sigma_\tau^2) \tag{76c}$$

where c is the speed of light. The clock drift noise, $n_\tau(t)$, is assumed to be a zero-mean, Gaussian white noise with power spectral density (PSD) equal to σ_τ^2. The velocity noise, $n_v(t)$, is also assumed to be zero-mean, white and Gaussian with PSD $\sigma_v^2 I$.

The measurements in this navigation system are taken at a rate of 1 Hz and the continuous-time system (76) is discretized accordingly. Assume that the GPS receiver tracks 4 satellites. Let $\delta r_k^{(i)}$ denote the GPS pseudorange error, defined as

$$\delta r_k^{(i)} \triangleq r_k^{(i)} - \rho_k^{(i)} \tag{77}$$

where $r_k^{(i)}$ is the range from the receiver's position, as computed by the INS, to the ith satellite, and $\rho_k^{(i)}$ is the pseudorange to the ith satellite as measured by the GPS receiver. Each of the GPS receiver channels may be subjected to a multipath measurement error, so that the pseudorange error is given by the following linear equation:

$$\delta r_k^{(i)} = a^{(i)T}\delta p_k + c\delta\tau_k + \gamma_k^{(i)} b_k^{(i)} + v_k^{(i)} \tag{78}$$

for $i = 1, 2, 3, 4$, where $a^{(i)}$ is the unit direction vector to the ith satellite, $v_k^{(i)} \sim \mathcal{N}(0, \sigma_r^2)$ is the pseudorange measurement white noise, $b_k^{(i)}$ is a multipath parameter and $\gamma_k^{(i)}$ indicates the presence of multipath error. For simplicity, the satellite constellation is assumed to be constant throughout the scenario with geometric dilution of precision (GDOP) equal to 2.13. The multipath parameters are assumed to behave as a stationary, zero-mean, Gaussian colored noise with autocorrelation

$$E\left[b_n^{(i)} b_m^{(i)}\right] = \Sigma_b^2 \alpha^{|n-m|} \quad 0 < \alpha < 1 \tag{79}$$

which corresponds to the following state–space model

$$b_{k+1}^{(i)} = \alpha b_k^{(i)} + w_{k+1}^{(i)} \quad b_0^{(i)} \sim \mathcal{N}\left(0, \Sigma_b^2\right) \tag{80}$$

where $w_{k+1}^{(i)} \sim \mathcal{N}(0, \sigma_b^2)$ is a zero-mean, white process noise satisfying

$$\sigma_b^2 = (1 - \alpha^2)\Sigma_b^2. \tag{81}$$

In addition to the GPS measurements the navigation system makes use of a terrain-following system, generating measurements modeled as:

$$\delta p_k^{\mathrm{TF}} = \delta p_k + \widetilde{v}_k \tag{82}$$

where $\widetilde{v}_k \sim \mathcal{N}(0, \sigma_{\mathrm{TF}}^2 I)$ is the measurement white noise. The terrain-following measurements are assumed to be less accurate than those of the GPS receiver. On the other hand, these measurements are assumed to be fault-free over the scenario duration. The numerical example parameters are summarized in Table I.

TABLE I
NUMERICAL STUDY PARAMETERS

Parameter	Value
Σ_p^2	$10^2 [\mathrm{m}^2]$
Σ_v^2	$5^2 [\mathrm{m}^2/\mathrm{sec}^2]$
Σ_τ^2	$10^2 [\mathrm{m}^2]$
σ_τ^2	$0.02^2 [\mathrm{m}^2/\mathrm{sec}]$
σ_r^2	$20^2 [\mathrm{m}^2]$
Σ_b^2	$50^2 [\mathrm{m}^2]$
σ_b^2	$5^2 [\mathrm{m}^2]$
σ_{TF}^2	$100^2 [\mathrm{m}^2]$
α	0.995

TABLE II
TEST CASE PARAMETERS

Test case	$\sigma_v^2 [\mathrm{m}^2/\mathrm{sec}^3]$	$P_{11}^{(i)}$	$P_{10}^{(i)}$
Nominal	1^2	0.99	0.01
High multipath probability	1^2	0.995	0.05
Low process noise	0.1^2	0.995	0.05

The newly proposed lower bound and the MCLB are used to assess the estimation performance of the following two filters.

1) The first filter utilizes the fault-free terrain-following measurements only. This filter serves as a baseline filter with a minimal measurement configuration. Notice, that this is the only minimal measurement configuration: the other potential minimal configuration, namely, the configuration utilizing the fault-prone GPS measurements only, renders the system described above not completely observable. Since the system with terrain-following measurements only is linear and Gaussian, the baseline filter is a standard KF.
2) The second filter is designed to examine the possible estimation accuracy contribution of the fault-prone GPS measurements, when used in addition to the baseline terrain-following measurements. Since the total system is hybrid, it is proposed to use the interacting multiple model (IMM) algorithm [14], which is known to be a powerful tool in filtering of hybrid systems.

1000 Monte-Carlo runs are used to evaluate the performance of the KF and 400 Monte-Carlo runs are used to evaluate the performance of the IMM filter. Three test cases, whose numerical parameters are summarized in Table II, are studied. In all three cases the initial fault probabilities, $\Pi_0^{(i)}$, are assumed to be zero. The position and velocity estimation performance metrics used in this study are defined as the root-sum-square (RSS) of the components of the position and velocity root mean square (RMS) estimation errors, respectively, namely

$$e_k \triangleq \sqrt{\frac{1}{M}\sum_{j=1}^{j=M}\left(e_x^{(j)}(k)^2 + e_y^{(j)}(k)^2 + e_z^{(j)}(k)^2\right)} \tag{83}$$

where $e_i^{(j)}(k), i = x, y, z$, is the estimation error in the position or velocity i component at the kth time step of the jth Monte-Carlo run, and M is the total number of Monte-Carlo runs.

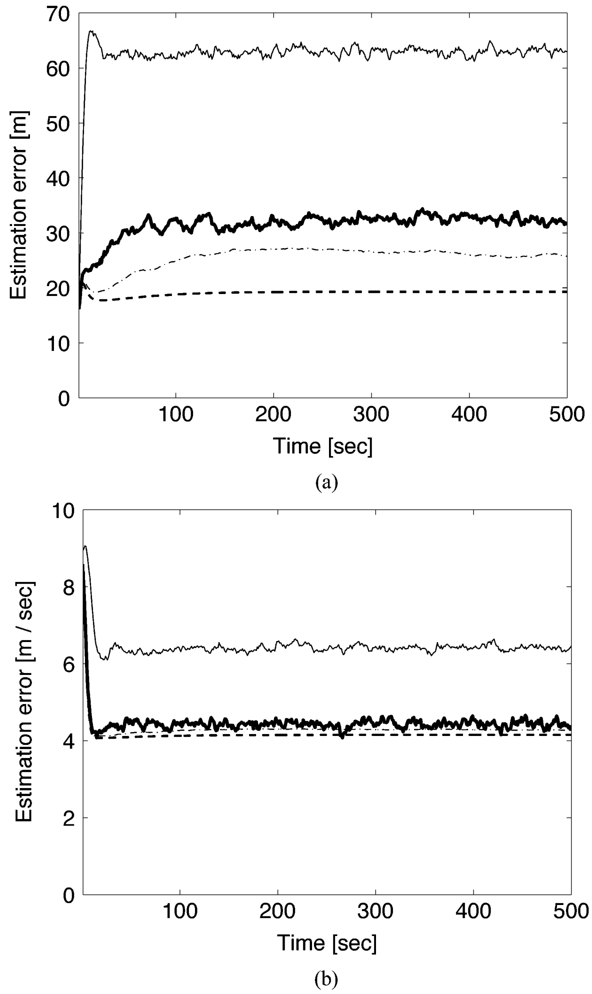

Fig. 1. Estimation errors achieved by IMM (bold solid line) and KF (thin solid line) vs the CRLB-type lower bound (thin dashed line) and the MCLB (thin dashed–dotted line) in the nominal case. (a) Position. (b) Velocity.

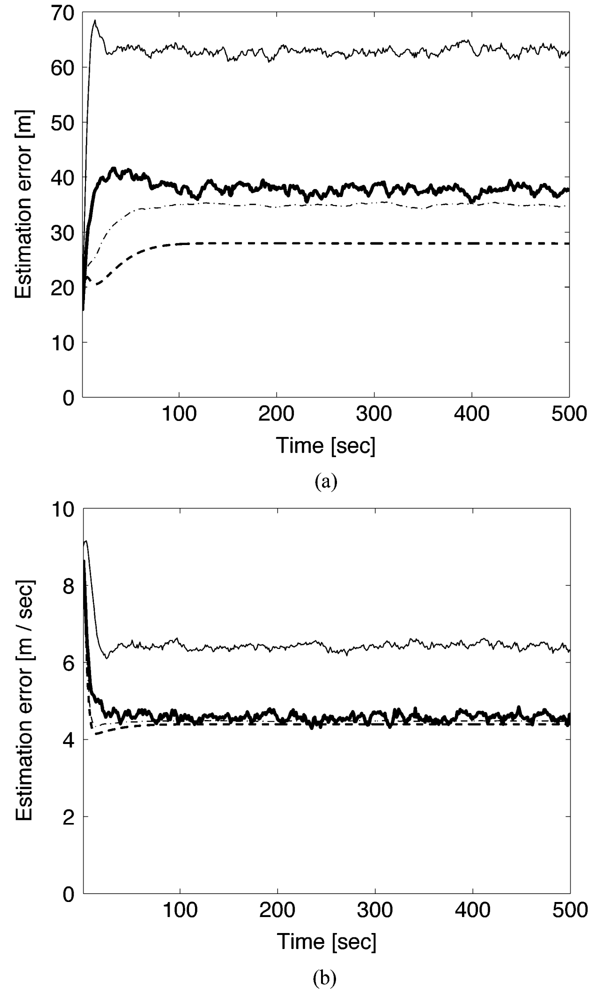

Fig. 2. Estimation errors achieved by IMM (bold solid line) and KF (thin solid line) vs the CRLB-type lower bound (thin dashed line) and the MCLB (thin dashed–dotted line) in the high multipath probability case. (a) Position. (b) Velocity.

The results of the first case study are presented in Fig. 1, that shows the position [Fig. 1(a)] and velocity [Fig. 1(b)] estimation errors. Comparing the absolute errors of the filters one can see that the IMM filter is about two times more accurate than the KF in position estimation [see Fig. 1(a)] and about 1.5 more accurate in velocity estimation [see Fig. 1(b)]. However, a direct comparison to the CRLB-type lower bound reveals that the IMM is much more efficient (in the statistical sense) than the KF: The position estimation error is about 3 times closer to the lower bound than the position estimation error of the KF. Considering the velocity estimates one can see that the IMM velocity error almost reaches the lower bound. This means that the IMM velocity estimates are almost optimal in the minimum variance sense, and also that the proposed lower bound is tight in this case.

The results of the second case are presented in Fig. 2. Due to the high multipath probability, the position error of the IMM filter and the corresponding CRLB-type lower bound increase [see Fig. 2(a)]. The KF performance remains about the same as in the previous case, since the KF estimates do not depend on the GPS measurements. Again, whereas its position estimate is better by only 70% than that of the KF, a direct comparison to the lower bound reveals that the IMM is 3.6 times more efficient than the KF, and its velocity estimate can be regarded as optimal (rendering the corresponding lower bound tight).

Examining the third case (see Fig. 3) one can notice that due to the low process noise the estimation errors of both filters, as well as the CRLB-type lower bounds, diminish. Now the velocity estimates of both filters can be considered as optimal [see Fig. 3(b)]. As for the position errors, the KF performance becomes closer to that of IMM filter relatively to the previous two cases: The IMM estimation error is only 1.2 times better than that of the KF and only 1.6 times closer to the lower bound.

Figs. 1–3 also show the MCLB of (69). 100 Monte-Carlo runs were used to evaluate this bound, which is associated with a computational load a hundred times larger than that of the proposed CRLB-type lower bound. One can see, especially in the position errors [Figs. 1(a), 2(a), and 3(a)], that the CRLB-type lower bound is indeed less tight than the MCLB, as has been predicted in Proposition 5.1.

VII. CONCLUSION

A CRLB-type lower bound for a class of systems with fault-prone measurements is presented. Lower bounds for both the state and the Markovian interruption variables (fault indicators) of the system are derived, based on the recently presented sequential version of the CRLB for general nonlinear systems. The

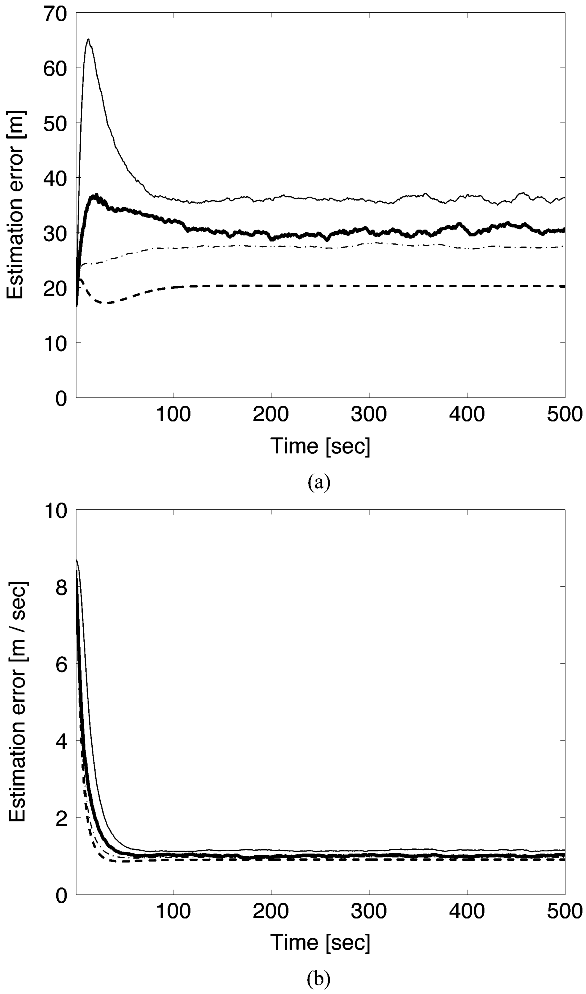

Fig. 3. Estimation errors achieved by IMM (bold solid line) and KF (thin solid line) vs the CRLB-type lower bound (thin dashed line) and the MCLB (thin dashed–dotted line) in the low process noise case. (a) Position. (b) Velocity.

derivation is based on an approximation of the discrete distribution of the interruption variables by a continuous one. The lower bound is then obtained via a limiting process applied to the approximating system.

The results presented in this paper facilitate a relatively simple calculation of a nontrivial CRLB-type lower bound for the state vector of systems with fault-prone measurements. Application of the CRLB-type lower bound to both the state vector and the measurement interruption variables does render the bound for the interruption process variables trivially zero, and shows that it is unable to take into account the effect of the fault vector estimation errors on the state vector estimation accuracy. However, an alternative, non-CRLB-type nontrivial lower bound for the interruption variables has been recently presented elsewhere by the authors.

The utility and applicability of the proposed lower bound are demonstrated via a numerical example involving a simple navigation system aided by a fault-prone GPS receiver and terrain-following position measurements. It is shown that the new lower bound serves as an efficient tool in the design of filters for this fault-prone system, as it facilitates the assessment of candidate filters, designed for different measurement system configurations.

APPENDIX A
AUXILIARY RESULTS

In this appendix, some auxiliary results are presented, that are used in the derivation and discussion of the lower bound (Sections IV and V). The proofs of the Lemmas are omitted for brevity, and can be found in [32].

Lemma A.1: Let $f_{\mu,R}(x)$ be a Gaussian pdf family with μ and R denoting the mean and covariance matrix of each of its members, respectively. Let μ_0, R_0 be some particular values of μ, R, with $R_0 > 0$. Then, for a function $g(x)$ that satisfies

$$\left|\int_{-\infty}^{+\infty} g(x) f_{\mu,R}(x)\mathrm{d}x\right| < \infty \tag{A.1a}$$

$$\left|\int_{-\infty}^{+\infty} g(x)^2 f_{\mu_0,R_0}(x)\mathrm{d}x\right| < \infty \tag{A.1b}$$

for all μ, R that belong to some neighborhood of μ_0, R_0, the following holds true:

$$\lim_{\mu\to\mu_0, R\to R_0} \int_{-\infty}^{+\infty} g(x) f_{\mu,R}(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} g(x) f_{\mu_0,R_0}(x)\,\mathrm{d}x. \tag{A.2}$$

Lemma A.2: Let $p_x(X)$ be the pdf of a random vector x. Let $A(X)$ be some positive–semidefinite matrix such that

$$\left|\int_{-\infty}^{+\infty} A_{ij}(X) p_x(X)\,\mathrm{d}X\right| < \infty. \tag{A.3}$$

Then, every element of $A(X)$ is absolutely integrable in the sense that

$$\int_{-\infty}^{+\infty} |A_{ij}(X)| p_x(X)\,\mathrm{d}X < \infty. \tag{A.4}$$

Lemma A.3: Let $G(x)$ be a function, continuous at some point $x = \mu$, that satisfies

$$\exists\sigma_0 > 0 \quad \text{s.t.}\ \frac{1}{\sqrt{2\pi}\sigma_0}\int_{-\infty}^{+\infty} |G(x)| e^{\frac{(x-\mu)^2}{2\sigma_0^2}}\,\mathrm{d}x = M < \infty. \tag{A.5}$$

Then

$$\lim_{\sigma\to 0^+} \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} G(x) e^{\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x = G(\mu). \tag{A.6}$$

Lemma A.4: Let X be a positive–definite random matrix with nonsingular mathematical expectation, and let A be some deterministic positive–semidefinite matrix of the same dimensions. Then

$$E[X^{-1}] \geqslant (E[X])^{-1} \tag{A.7a}$$

$$E[(X^{-1}+A)^{-1}] \leqslant ((E[X])^{-1}+A)^{-1}. \tag{A.7b}$$

APPENDIX B
REGULARITY CONDITIONS

The regularity conditions for the system defined by (1), (3), and (15) are that the first two derivatives of the joint pdf $p_{\mathcal{Y}_k,x_1,\ldots,x_k,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_k}(\Upsilon_k, X_1,\ldots,X_k,\Gamma_1,\ldots,\Gamma_k)$ with respect to X_l's and Γ_l's are absolutely integrable, and that the first moments of the estimation errors are finite (see [17, p. 72]). Notice that

$$\begin{aligned} & p_{\mathcal{Y}_k, x_1,\ldots,x_k,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_k}(\Upsilon_k, X_1,\ldots,X_k,\Gamma_1,\ldots,\Gamma_k) \\ & \quad = p_{\mathcal{Y}_k \,|\, x_1,\ldots,x_k,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_k}(\Upsilon_k \,|\, X_1,\ldots,X_k,\Gamma_1,\ldots,\Gamma_k) \\ & \qquad \times p_{x_1,\ldots,x_k}(X_1,\ldots,X_k) p_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_k}(\Gamma_1,\ldots,\Gamma_k). \quad \text{(B.1)} \end{aligned}$$

Now, all the pdfs in (B.1) are either Gaussian or linear combinations of Gaussian distributions with coefficients depending on $P^{(i)}(\Gamma)$'s. Recalling that the functions $P^{(i)}(\Gamma)$'s are twice differentiable with bounded derivatives yields absolute integrability of the joint pdf $p_{\mathcal{Y}_k, x_1,\ldots,x_k,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_k}(\Upsilon_k, X_1,\ldots,X_k,\Gamma_1,\ldots,\Gamma_k)$.

As for the requirement that the first moments of the estimation errors be finite, the discussion is restricted to those estimators that produce finite second moments of the estimation error. Therefore, this requirement is naturally satisfied.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewer for constructive suggestions and comments.

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