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# Minimal-Parameter Attitude Matrix Estimation from Vector Observations

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**A computationally efficient, sequential method is presented for attitude matrix estimation using gyro and vector measurements. The method is based on a recently introduced, minimal-parameter third-order method for solving the orthogonal matrix differential equation in $\mathbb{R}^n$. In the three-dimensional case, these third-order attitude parameters can be interpreted as temporal integrals of the body-frame angular velocity components. A nonlinear algorithm is developed, which uses this minimal set of three parameters to estimate the nine-parameter direction-cosine matrix. Having an extremely simple kinematic equation, these parameters render the resulting estimator highly computationally efficient. An orthogonalization procedure, incorporated into the measurement processing stage, enhances the accuracy and stability of the resulting algorithm, yet retains reasonable simplicity. The performance of the new estimator is demonstrated via a numerical simulation study.**

## Introduction

USING a sequence of vector measurements for attitude determination has been intensively investigated over the last three decades. First proposed in 1965 by Wahba,[1] the problem is to estimate the attitude of a spacecraft based on a sequence of noisy vector observations, resolved in the body-fixed coordinate system and in a reference system. Body-fixed vector observations are typically obtained from onboard sensors, such as star trackers, sun sensors, or magnetometers. Corresponding reference observations are obtained by using an ephemeris routine (for a sun observation), from orbit data and a magnetic field routine (for a magnetic field observation), or from a star catalog (for star observations). Uses of attitude determination from vector observations were reported in Refs. 2 and 3.

Inertial reference systems typically utilize vector measurements in combination with strap-down gyros to estimate both the spacecraft attitude and the gyro drift rate biases. Several approaches have been proposed for the design of such systems, differing mainly in their choice of attitude representation method.

The quaternion, a popular rotation specifier, was used in Refs. 4 and 5, in the framework of extended Kalman filtering (EKF) algorithms. The incorporation of the QUEST measurement model within a Kalman filter's measurement update stage was presented in Ref. 6. In Ref. 7, vector observations were used to estimate both the quaternion and the angular velocity of the spacecraft, in a gyroless attitude determination and control setting. The main advantage of using the quaternion representation is that it is not singular for any rotation. Moreover, its kinematic equation is linear and the computation of the associated attitude matrix involves only algebraic expressions. However, the quaternion representation is not minimal because it is four dimensional. This leads to a normalization constraint that has to be addressed in filtering algorithms and increases the associated computation load.

Euler angles were used by only a few researchers[8,9] to parameterize the attitude in the context of gyro-vector measurements attitude estimation. Although this parameterization is minimal, its use imposes a large computational burden due to the transcendental expressions involved in the computation of the attitude matrix. Moreover, Euler angles are singular, as are all three-dimensional attitude representations.

In a recent effort to alleviate the computational burden, an EKF attitude estimator was presented[10] that utilized the Rodrigues parameters (also known as the Gibbs vector). Being a minimal set of attitude parameters, the choice of this parameterization renders the resulting estimator computationally efficient; however, the Rodrigues parameters are singular for 180-deg rotations. The modified Rodrigues parameters (MRP), on the other hand, allow rotations up to 360 deg (Ref. 11). Using this observation, an MRP-based estimator has recently been presented.[12]

The direction-cosine matrix (DCM), a natural attitude representation, was used in a gyro-star tracker setting by several researchers. Because it is inherently nonsingular, it requires no special singularity-handling procedures. Moreover, its kinematic equation is linear, as is its associated vector measurement equation, which greatly facilitate the filter implementation. A recursive, EKF-type DCM identification algorithm was introduced by Bar-Itzhack and Reiner.[13] Although the advantages of directly parameterizing the attitude using the DCM are clear, the main disadvantage of this approach is computational, as it requires the estimation of a nine-dimensional parameter vector.

The work presented in this paper proposes to sequentially estimate the attitude matrix using a minimal-dimension filter, thus alleviating the computational burden normally associated with DCM identification. It is assumed, as usual, that the body-referenced angular velocity is measured by an orthogonal triad of rate gyros. The approach taken to this end is motivated by the idea of finding a minimal-parameter solution to the orthogonal matrix differential equation in $\mathbb{R}^n$, first introduced by Bar-Itzhack and Markley.[14] In this recent work targeted at solving a problem first raised in Ref. 15, they presented a minimal-parameter solution to the orthogonal matrix differential equation

$$\dot{V}(t) = W(t)V(t)$$

$$V(t) \in \mathbb{R}^{n,n}, \qquad W(t) = -W^T(t) \quad \forall t \geq t_0 \tag{1a}$$

$$V(t_0) = V_0, \qquad V_0 V_0^T = I \tag{1b}$$

where the overdot indicates the temporal derivative. Exploiting the properties of $V$ and $W$, Bar-Itzhack and Markley introduced a novel minimal parameterization of the orthogonal matrix $V$. Based on extended Rodrigues parameters, this parameterization enabled solving Eq. (1) using only $n(n-1)/2$ parameters, as opposed to $n^2$ integrations implied by a straightforward solution of Eq. (1).

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Motivated by Ref. 14, Ronen and Oshman[16] have recently introduced a third-order method for the solution of the orthogonal matrix differential equation in $\mathbb{R}^n$. The method is based on a third-order, minimal parameterization of the orthogonal matrix $V$ using the $n(n-1)/2$ off-diagonal terms of the skew-symmetric matrix

$$A(t, t_0) \triangleq \int_{t_0}^{t} W(\tau)\,\mathrm{d}\tau$$

For the three-dimensional case, these parameters, hereafter called integrated rate parameters (IRP), are the angles resulting from time integration of the body-frame components of the spacecraft angular velocity.

The idea underlying the work presented herein is to utilize the minimal, three-dimensional IRP vector to estimate the nine-parameter attitude matrix. Building on this state vector, the resulting three-dimensional filter possesses an extremely simple time-propagation procedure, which is at the heart of its computational efficiency. The DCM orthogonality constraint is also addressed, contributing to the filter's accuracy and numerical stability, yet keeping its structure reasonably simple.

The following section briefly reviews the IRP minimal-parameter, third-order method for the solution of the attitude kinematic equation. The filtering, orthogonalization, and prediction stages of the attitude estimator are developed next. Special attention is given to the analysis of the potential effects of the orthogonalization step on the estimator's structure. In the next section a numerical simulation study is presented that demonstrates the accuracy and robustness of the new algorithm. Concluding remarks are offered in the last section.

## Third-Order, Minimal Attitude Parameters

For completeness, this section briefly reviews the minimal-parameter problem and the IRP method. Then, the method is adapted to the three-dimensional attitude kinematic equation.

### Minimal-Parameter Problem

Given the matrix differential equation (1), the minimal-parameter problem is to find 1) a set of $m = n(n-1)/2$ parameters that unambiguously define $V(t)$, 2) the differential equation satisfied by these parameters, 3) the transformation that maps these parameters into the matrix $V$, and 4) a simple and efficient method to solve the parameters' differential equation and to compute $V(t)$.

The third-order method for the solution of Eq. (1), recently introduced by Ronen and Oshman,[16] is summarized in the following.

Let the skew-symmetric matrix $A(t, t_0)$ be defined as

$$A(t, t_0) \triangleq \int_{t_0}^{t} W(\tau)\,\mathrm{d}\tau \tag{2}$$

Then, a third-order approximation of the solution $V(t)$ in terms of the entries of the matrix $A(t, t_0)$ is given by the matrix $\tilde{V}(t, t_0)$, defined as

$$\tilde{V}(t, t_0) \triangleq \left\{ I + A(t, t_0) + \frac{A^2(t, t_0)}{2!} + \frac{A^3(t, t_0)}{3!} + \frac{(t - t_0)}{3!}[A(t, t_0)W_0 - W_0 A(t, t_0)] \right\} V_0 \tag{3}$$

where $W_0 = W(t_0)$. Moreover, $\tilde{V}$ is a third-order approximation of an orthogonal matrix, in the sense that

$$\tilde{V}(t, t_0)\tilde{V}^T(t, t_0) = I + \mathcal{O}\left[(t - t_0)^4\right] \tag{4}$$

where $\mathcal{O}(x)$ denotes a function of $x$ that has the property that $\mathcal{O}(x)/x$ is bounded as $x \to 0$.

Referring now to the minimal-parameter problem, the new parameters, which define the third-order solution of Eq. (1), are the $n(n-1)/2$ off-diagonal terms of $A(t, t_0)$. For the three-dimensional case, these parameters have a simple geometric interpretation: they are the angles resulting from a temporal integration of the three components of the angular velocity vector

$$\boldsymbol{\omega}(t) \triangleq [\omega_1(t) \quad \omega_2(t) \quad \omega_3(t)]^T \tag{5}$$

where $\omega_i$ is the angular velocity component along the $i$ axis of the initial coordinate system, and $i = 1, 2, 3$ for $x, y, z$, respectively.

The differential equation satisfied by these parameters is

$$\dot{A}(t, t_0) = W(t), \qquad A(t_0, t_0) = 0 \tag{6}$$

which can be easily solved using any quadrature scheme. As demonstrated in Ref. 16, the new minimal-parameter method is both computationally efficient and accurate.

### Attitude Matrix Kinematic Equation

In the three-dimensional case, the orthogonal matrix referred to is the attitude matrix, or the direction cosine matrix (DCM), denoted by $D(t)$. The differential equation satisfied by this matrix is the well-known equation

$$\dot{D}(t) = \Omega(t)D(t), \qquad D(t_0) = D_0 \tag{7}$$

where $\Omega(t) = -[\boldsymbol{\omega}(t)\times]$, the cross product matrix corresponding to $\boldsymbol{\omega}(t)$, is defined according to

$$[\boldsymbol{a}\times] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \qquad \forall \boldsymbol{a} \in \mathbb{R}^3 \tag{8}$$

This notation reflects the fact that

$$[\boldsymbol{a}\times]\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}, \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \tag{9}$$

In this case, the matrix $A(t, t_0)$ takes the form

$$A(t, t_0) \triangleq -[\boldsymbol{\theta}(t)\times] \tag{10}$$

where the parameter vector $\boldsymbol{\theta}(t)$ is defined as

$$\boldsymbol{\theta}(t) \triangleq [\theta_1(t) \quad \theta_2(t) \quad \theta_3(t)]^T \tag{11}$$

and

$$\theta_i(t) \triangleq \int_{t_0}^{t} \omega_i(\tau)\,\mathrm{d}\tau, \qquad i = 1, 2, 3 \tag{12}$$

## Attitude Estimator

In this section we develop the attitude estimation algorithm. The development of the algorithm relies on the choice of the parameter vector $\boldsymbol{\theta}$, defined in Eqs. (11) and (12), to be the estimator's state vector.

Let the sampling period be denoted by $T \triangleq t_{k+1} - t_k$. Using the notation $\boldsymbol{\theta}(k) \triangleq \boldsymbol{\theta}(t_k)$, the state vector at time $t_k$ is

$$\boldsymbol{\theta}(k) = [\theta_1(k) \quad \theta_2(k) \quad \theta_3(k)]^T \tag{13}$$

and Eq. (12) implies

$$\theta_i(k) = \int_{t_0}^{t_k} \omega_i(\tau)\,\mathrm{d}\tau, \qquad i = 1, 2, 3 \tag{14}$$

where $\boldsymbol{\omega}(t)$ is the spacecraft angular velocity vector, defined in Eq. (5). From Eq. (14) we have

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k) + \int_{t_k}^{t_{k+1}} \boldsymbol{\omega}(\tau)\,\mathrm{d}\tau \tag{15}$$

Defining $A(k+1, k)$ to be the discrete-time analog of $A(t, t_0)$, i.e.,

$$A(k+1, k) \triangleq -[[\boldsymbol{\theta}(k+1) - \boldsymbol{\theta}(k)]\times] \tag{16}$$

Eq. (3) is rewritten as

$$D(k+1) = \left\{ I + A(k+1, k) + \tfrac{1}{2}A^2(k+1, k) + \tfrac{1}{6}A^3(k+1, k) + \tfrac{1}{6}T[A(k+1, k)\Omega(k) - \Omega(k)A(k+1, k)] \right\} D(k) \tag{17}$$

In practice we only have access to the measured angular velocity, denoted by $\hat{\omega}(t)$, which satisfies

$$\hat{\omega}(t) = \omega(t) + \delta\omega(t) \tag{18}$$

Here $\delta\omega(t)$ is the rate-gyro (RG) measurement noise. For simplicity, this noise is assumed to be a zero-mean, white, Gaussian process, denoted as

$$\delta\omega(t) \sim \mathcal{WN}[0, Q(t)] \tag{19}$$

where its intensity $Q(t)$ (the power spectral density matrix) is defined by

$$E\{\delta\omega(t)\delta\omega^T(s)\} = Q(t)\delta(t-s) \tag{20}$$

(The incorporation of more elaborate gyro noise models is straightforward.[17])

The estimation algorithm comprises three subalgorithms. In the filtering stage, the state estimate and the estimation error covariance matrix are updated across the newly acquired measurement. Following the filtering stage, the estimated attitude matrix is orthogonalized, to enhance the algorithm stability by annihilating the numerical errors that have accumulated during the recent prediction and filtering stages. The prediction stage deals with the propagation in time of these variables between consecutive measurement updates. These three procedures are developed in the ensuing.

## Filtering

Let the minimum mean-squared estimate (MMSE) of $\boldsymbol{\theta}(j)$ based on measurements up to and including $t_k$ be denoted by $\hat{\boldsymbol{\theta}}(j \mid k)$. Assume that at $t_{k+1}$ we have on hand the predicted parameter vector $\hat{\boldsymbol{\theta}}(k+1 \mid k)$ and its corresponding prediction error covariance matrix $P(k+1 \mid k) \triangleq E\{\tilde{\boldsymbol{\theta}}(k+1 \mid k)\tilde{\boldsymbol{\theta}}^T(k+1 \mid k)\}$, where the estimation error is defined as

$$\tilde{\boldsymbol{\theta}}(j \mid k) \triangleq \hat{\boldsymbol{\theta}}(j) - \hat{\boldsymbol{\theta}}(j \mid k) \tag{21}$$

The purpose of the filtering scheme, to be developed in the sequel, is to compute the a posteriori estimate and the corresponding error covariance matrix by way of incorporating the new vector measurements acquired at $t_{k+1}$.

As the first step in developing the measurement update algorithm, we next derive the observation equation, relating the acquired vector measurements to the state.

### *Observation Equation*

Let $\mathcal{S}_u$ and $\mathcal{S}_v$ denote the reference Cartesian coordinate system and the body-fixed Cartesian coordinate system, respectively. The new pair of corresponding noisy vector measurements consists of the unit vectors $\boldsymbol{u}(k+1)$ and $\boldsymbol{v}(k+1)$, which represent the measured values of the same vector $\boldsymbol{r}(k+1)$, resolved in $\mathcal{S}_u$ and in $\mathcal{S}_v$, respectively. The direction-cosine matrix $D(k+1)$, representing the true attitude of $\mathcal{S}_v$ relative to $\mathcal{S}_u$ at time $t_{k+1}$, transforms the true vector representation $\boldsymbol{u}_0$ in $\mathcal{S}_u$ into its corresponding true representation $\boldsymbol{v}_0$ in $\mathcal{S}_v$ according to

$$\boldsymbol{v}_0(k+1) = D(k+1)\boldsymbol{u}_0(k+1) \tag{22}$$

Assuming no constraint on the measurement noise direction, the body-frame measured unit vector $\boldsymbol{v}(k+1)$ is related to the true vector according to

$$\boldsymbol{v}(k+1) = \frac{\boldsymbol{v}_0(k+1) + \boldsymbol{n}'_v(k+1)}{\|\boldsymbol{v}_0(k+1) + \boldsymbol{n}'_v(k+1)\|} \tag{23}$$

where the sensor measurement noise $n'_v$ is a white, Gaussian noise sequence with

$$\boldsymbol{n}'_v(k+1) \sim \mathcal{N}[0, R'_v(k+1)] \tag{24}$$

Because both $\boldsymbol{v}_0(k+1)$ and $\boldsymbol{v}(k+1)$ are unit vectors, it follows from Eq. (23) that

$$\boldsymbol{v}(k+1) = \boldsymbol{v}_0(k+1) + \mathcal{P}^{\perp}_{v_0}(k+1)\boldsymbol{n}'_v(k+1) + \mathcal{O}\left(\|\boldsymbol{n}'_v(k+1)\|^2\right) \tag{25}$$

where the idempotent matrix

$$\mathcal{P}^{\perp}_{v_0}(k+1) \triangleq I - \boldsymbol{v}_0(k+1)\boldsymbol{v}_0^T(k+1) \tag{26}$$

is the orthogonal projector onto the orthogonal complement of span $\{\boldsymbol{v}_0(k+1)\}$. Defining, therefore, the effective measurement noise associated with the measurement $\boldsymbol{v}(k+1)$ as

$$\boldsymbol{n}_v(k+1) \triangleq \mathcal{P}^{\perp}_{v_0}(k+1)\boldsymbol{n}'_v(k+1) \tag{27}$$

yields the following measurement model:

$$\boldsymbol{v}(k+1) = \boldsymbol{v}_0(k+1) + \boldsymbol{n}_v(k+1) \tag{28}$$

where the effective measurement noise is, to a good approximation, a white, Gaussian sequence with

$$\boldsymbol{n}_v(k+1) \sim \mathcal{N}(0, R_v(k+1)) \tag{29}$$

and the measurement noise covariance matrix is

$$R_v(k+1) \triangleq \mathcal{P}^{\perp}_{v_0}(k+1)R'_v(k+1)\mathcal{P}^{\perp}_{v_0}(k+1) \tag{30}$$

*Remark 1.* This measurement model is similar to that derived in Ref. 6 for complete vector sensors.

*Remark 2.* In practice, because $\boldsymbol{v}_0(k+1)$ is not known, the projector matrix $\mathcal{P}^{\perp}_{v_0}(k+1)$ can be approximated using the measured value $\boldsymbol{v}(k+1)$.

*Remark 3.* If, in the particular sensor used, $\boldsymbol{n}'_v$ is constrained to be orthogonal to $\boldsymbol{v}_0$, then Eq. (30) reduces to

$$R_v(k+1) = R'_v(k+1) \tag{31}$$

Furthermore, if the measurement noise is isotropic, $R'_v(k+1) = \sigma^2 I$, then $R_v(k+1) = \sigma^2\mathcal{P}^{\perp}_{v_0}(k+1)$.

The vector measurements relative to the reference coordinate system are commonly assumed to be accurately known. However, to account for nonideal effects (e.g., star catalog errors), it is assumed in this work that the true unit vector and the measured unit vector are related according to

$$\boldsymbol{u}(k+1) = \boldsymbol{u}_0(k+1) + \boldsymbol{n}_u(k+1) \tag{32}$$

where $\boldsymbol{n}_u \perp \boldsymbol{u}_0$ is a white, Gaussian measurement noise that is uncorrelated with $\boldsymbol{n}_v$ and satisfies

$$\boldsymbol{n}_u(k) \sim \mathcal{N}[0, R_u(k)] \tag{33}$$

with $R_u(k)$ being a known covariance matrix.

Because it is desired to relate the information contained in the measurements to the state vector at the corresponding time point, Eq. (22) is rewritten as

$$\boldsymbol{v}_0(k+1) = D[\boldsymbol{\theta}(k+1) - \boldsymbol{\theta}(k), D(k)]\boldsymbol{u}_0(k+1) \tag{34}$$

where the notation $D[\boldsymbol{\theta}(k+1) - \boldsymbol{\theta}(k), D(k)]$ reflects the fact that the attitude at time $t_{k+1}$ is related to the attitude at time $t_k$ via the IRP vector difference $\boldsymbol{\theta}(k+1) - \boldsymbol{\theta}(k)$ [see Eqs. (16) and (17)].

To exploit the information contained in the new vector measurements, the nonlinear measurement equation (34) is linearized about a nominal state, consisting of the most recent state estimate. Assuming that immediately after the previous measurement update (at $t_k$) linearization has been carried out about the a posteriori state estimate, the resulting nominal state at the current measurement update is the predicted estimate, $\hat{\boldsymbol{\theta}}(k+1 \mid k)$. Therefore, the predicted parameters are assumed to be related to the true ones according to

$$\boldsymbol{\theta}(k+1) = \hat{\boldsymbol{\theta}}(k+1 \mid k) + \delta\boldsymbol{\theta}(k+1) \tag{35}$$

where $\delta\boldsymbol{\theta}(k+1)$ is the perturbation of the parameter vector about the nominal, i.e., predicted, state. Using now the most recent estimates for $D(k)$ and $\boldsymbol{\theta}(k)$, namely, $\hat{D}(k \mid k)$ and $\hat{\boldsymbol{\theta}}(k \mid k)$, respectively, in Eq. (34), it follows from Eqs. (28), (32), and (35) that

$$\begin{aligned} \boldsymbol{v}(k+1) - \boldsymbol{n}_v(k+1) = D[&\hat{\boldsymbol{\theta}}(k+1 \mid k) + \delta\boldsymbol{\theta}(k+1) \\ &- \hat{\boldsymbol{\theta}}(k \mid k), \hat{D}(k \mid k)](\boldsymbol{u}(k+1) - \boldsymbol{n}_u(k+1)) \end{aligned} \tag{36}$$

However, as shown in the sequel, the a posteriori state estimate is zeroed after each measurement update (due to full reset control of the state). Hence, we should use the reset value of the state estimate,

$$\hat{\boldsymbol{\theta}}^c(k\,|\,k) = 0 \tag{37}$$

in Eq. (36). This allows us to rewrite Eq. (36) as

$$\boldsymbol{v}(k+1) - \boldsymbol{n}_v(k+1) = D[\hat{\boldsymbol{\theta}}(k+1\,|\,k) + \delta\boldsymbol{\theta}(k+1),\ \hat{D}(k\,|\,k)] \times [\boldsymbol{u}(k+1) - \boldsymbol{n}_u(k+1)] \tag{38}$$

where it is understood that Eq. (37) is used hereinafter in the computation of the third-order approximation for the attitude matrix. Now expand $D$ about the nominal parameter vector using a first-order Taylor series expansion, i.e.,

$$D[\hat{\boldsymbol{\theta}}(k+1\,|\,k) + \delta\boldsymbol{\theta}(k+1),\ \hat{D}(k\,|\,k)] = D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)] + \sum_{i=1}^{3} \frac{\partial}{\partial\boldsymbol{\theta}_i} D[\boldsymbol{\theta}(k+1),\ \hat{D}(k\,|\,k)]|_{\hat{\boldsymbol{\theta}}(k+1\,|\,k)} \delta\boldsymbol{\theta}_i(k+1) \tag{39}$$

where $()|_{\hat{\boldsymbol{\theta}}(k+1\,|\,k)}$ denotes "evaluated at $\hat{\boldsymbol{\theta}}(k+1\,|\,k)$." Using Eq. (17), the sensitivity matrices appearing in Eq. (39) are computed as

$$\frac{\partial}{\partial\boldsymbol{\theta}_i} D[\boldsymbol{\theta}(k+1),\ \hat{D}(k\,|\,k)]|_{\hat{\boldsymbol{\theta}}(k+1\,|\,k)} = G_i[\hat{\boldsymbol{\theta}}(k+1\,|\,k)]\hat{D}(k\,|\,k) \qquad i = 1, 2, 3 \tag{40}$$

where the matrices $\{G_i\}_{i=1}^3$ are

$$G_1 = \begin{bmatrix} 0 & \frac{1}{2}\boldsymbol{\theta}_2 - \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_3 - \frac{1}{6}T\boldsymbol{\omega}_2 & \frac{1}{2}\boldsymbol{\theta}_3 + \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_2 - \frac{1}{6}T\boldsymbol{\omega}_3 \\ \frac{1}{2}\boldsymbol{\theta}_2 + \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_3 + \frac{1}{6}T\boldsymbol{\omega}_2 & -\boldsymbol{\theta}_1 & 1 - \frac{1}{6}\left(\boldsymbol{\theta}_2^2 + \boldsymbol{\theta}_3^2\right) - \frac{1}{2}\boldsymbol{\theta}_1^2 \\ \frac{1}{2}\boldsymbol{\theta}_3 - \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_2 + \frac{1}{6}T\boldsymbol{\omega}_3 & -1 + \frac{1}{6}\left(\boldsymbol{\theta}_2^2 + \boldsymbol{\theta}_3^2\right) + \frac{1}{2}\boldsymbol{\theta}_1^2 & -\boldsymbol{\theta}_1 \end{bmatrix} \tag{41a}$$

$$G_2 = \begin{bmatrix} -\boldsymbol{\theta}_2 & \frac{1}{2}\boldsymbol{\theta}_1 - \frac{1}{3}\boldsymbol{\theta}_2\boldsymbol{\theta}_3 + \frac{1}{6}T\boldsymbol{\omega}_1 & -1 + \frac{1}{6}\left(\boldsymbol{\theta}_1^2 + \boldsymbol{\theta}_3^2\right) + \frac{1}{2}\boldsymbol{\theta}_2^2 \\ \frac{1}{2}\boldsymbol{\theta}_1 + \frac{1}{3}\boldsymbol{\theta}_2\boldsymbol{\theta}_3 - \frac{1}{6}T\boldsymbol{\omega}_1 & 0 & \frac{1}{2}\boldsymbol{\theta}_3 - \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_2 - \frac{1}{6}T\boldsymbol{\omega}_3 \\ 1 - \frac{1}{6}\left(\boldsymbol{\theta}_1^2 + \boldsymbol{\theta}_3^2\right) - \frac{1}{2}\boldsymbol{\theta}_2^2 & \frac{1}{2}\boldsymbol{\theta}_3 + \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_2 + \frac{1}{6}T\boldsymbol{\omega}_3 & -\boldsymbol{\theta}_2 \end{bmatrix} \tag{41b}$$

$$G_3 = \begin{bmatrix} -\boldsymbol{\theta}_3 & 1 - \frac{1}{6}\left(\boldsymbol{\theta}_1^2 + \boldsymbol{\theta}_2^2\right) - \frac{1}{2}\boldsymbol{\theta}_3^2 & \frac{1}{2}\boldsymbol{\theta}_1 + \frac{1}{3}\boldsymbol{\theta}_2\boldsymbol{\theta}_3 + \frac{1}{6}T\boldsymbol{\omega}_1 \\ -1 + \frac{1}{6}\left(\boldsymbol{\theta}_1^2 + \boldsymbol{\theta}_2^2\right) + \frac{1}{2}\boldsymbol{\theta}_3^2 & -\boldsymbol{\theta}_3 & \frac{1}{2}\boldsymbol{\theta}_2 - \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_3 + \frac{1}{6}T\boldsymbol{\omega}_2 \\ \frac{1}{2}\boldsymbol{\theta}_1 - \frac{1}{3}\boldsymbol{\theta}_2\boldsymbol{\theta}_3 - \frac{1}{6}T\boldsymbol{\omega}_1 & \frac{1}{2}\boldsymbol{\theta}_2 + \frac{1}{3}\boldsymbol{\theta}_1\boldsymbol{\theta}_3 - \frac{1}{6}T\boldsymbol{\omega}_2 & 0 \end{bmatrix} \tag{41c}$$

In Eqs. (41) the components of $\boldsymbol{\omega}$ are evaluated at $t_k$.

*Remark 4.* In a typical application, it can be assumed that the parameters $\{\boldsymbol{\theta}_i\}_{i=1}^3$ are small, such that the second-order quantities $\{\boldsymbol{\theta}_i\boldsymbol{\theta}_j\}_{i,j=1}^3$ are negligible in Eqs. (41). Using this small-angle approximation results in much simpler forms for $G_i[\hat{\boldsymbol{\theta}}(k+1\,|\,k)]$. The actual use of either Eqs. (41) or their small-angle approximation depends, in practice, on the dynamics of the specific application.

*Remark 5.* Notice the explicit dependence of the sensitivity matrices on the angular velocity, which sets this formulation apart from previous estimators using vector observations.[4,5,9,10]

Using now Eq. (39) in Eq. (38) and neglecting second-order terms yields

$$\boldsymbol{v}(k+1) - D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)]\boldsymbol{u}(k+1) = \sum_{i=1}^{3} G_i[\hat{\boldsymbol{\theta}}(k+1\,|\,k)]\hat{D}(k\,|\,k)\delta\boldsymbol{\theta}_i(k+1)\boldsymbol{u}(k+1) - D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)]\boldsymbol{n}_u(k+1) + \boldsymbol{n}_v(k+1) \tag{42}$$

Observe that the first member in the right-hand side (RHS) of Eq. (42) can be recast as

$$\sum_{i=1}^{3} G_i[\hat{\boldsymbol{\theta}}(k+1\,|\,k)]\hat{D}(k\,|\,k)\delta\boldsymbol{\theta}_i(k+1)\boldsymbol{u}(k+1) = H(k+1)\delta\boldsymbol{\theta}(k+1) \tag{43}$$

where the columns of the (observation) matrix

$$H(k+1) \equiv [\boldsymbol{h}_1(k+1) \quad \boldsymbol{h}_2(k+1) \quad \boldsymbol{h}_3(k+1)] \in \mathbb{R}^{3,3} \tag{44}$$

are

$$\boldsymbol{h}_i(k+1) = G_i[\hat{\boldsymbol{\theta}}(k+1\,|\,k)]\hat{D}(k\,|\,k)\boldsymbol{u}(k+1), \qquad i = 1, 2, 3 \tag{45}$$

Define now the effective measurement $\boldsymbol{y}(k+1)$ to be

$$\boldsymbol{y}(k+1) \triangleq \boldsymbol{v}(k+1) - D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)]\boldsymbol{u}(k+1) \tag{46}$$

and the effective measurement noise to be

$$\boldsymbol{n}(k+1) \triangleq \boldsymbol{n}_v(k+1) - D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)]\boldsymbol{n}_u(k+1) \tag{47}$$

Then, using these definitions in Eq. (42) yields the following measurement equation:

$$\boldsymbol{y}(k+1) = H(k+1)\delta\boldsymbol{\theta}(k+1) + \boldsymbol{n}(k+1) \tag{48}$$

The measurement noise is a white, Gaussian sequence with

$$\boldsymbol{n}(k+1) \sim \mathcal{N}[0,\ R(k+1)] \tag{49}$$

where

$$R(k+1) \triangleq R_v(k+1) + D[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)] \times R_u(k+1)D^T[\hat{\boldsymbol{\theta}}(k+1\,|\,k),\ \hat{D}(k\,|\,k)] \tag{50}$$

Having the linearized measurement equation (48) and the statistical characterization of the measurement noise (49) on hand, we can now derive the MMSE estimator for the parameter vector.

### *State and Covariance Update*

Using Eqs. (21) and (35), we have

$$\delta\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}(k+1) - \hat{\boldsymbol{\theta}}(k+1\,|\,k) = \tilde{\boldsymbol{\theta}}(k+1\,|\,k) \tag{51}$$

Because $\hat{\boldsymbol{\theta}}(k+1\,|\,k)$ is an unbiased, MMSE predictor, we have

$$E\{\delta\boldsymbol{\theta}(k+1)\} = E\{\tilde{\boldsymbol{\theta}}(k+1\,|\,k)\} = 0 \tag{52}$$

and

$$\mathrm{cov}\{\delta\boldsymbol{\theta}(k+1)\} = \mathrm{cov}\{\tilde{\boldsymbol{\theta}}(k+1\,|\,k)\} = P(k+1\,|\,k) \tag{53}$$

Hence,

$$\delta\boldsymbol{\theta}(k+1) \sim \mathcal{N}(0,\ P(k+1\,|\,k)) \tag{54}$$

Using the linearized measurement equation (48) and the statistical properties of the measurement and prediction errors, Eqs. (49) and (54), respectively, the MMSE estimator of $\delta\boldsymbol{\theta}(k+1)$ is[18]

$$\widehat{\delta\boldsymbol{\theta}}(k+1\,|\,k+1) = K(k+1)\boldsymbol{y}(k+1) \tag{55}$$

where $K(k+1)$, the estimator gain matrix, is computed as

$$K(k+1) = P(k+1\,|\,k)H^T(k+1) \times [H(k+1)P(k+1\,|\,k)H^T(k+1) + R(k+1)]^{-1} \tag{56}$$

Also, from Eq. (51) we have

$$\widehat{\delta\theta}(k+1\,|\,k+1) = \hat{\theta}(k+1\,|\,k+1) - \hat{\theta}(k+1\,|\,k) \tag{57}$$

Using Eq. (57) in Eq. (55) finally yields the state measurement update equation

$$\hat{\theta}(k+1\,|\,k+1) = \hat{\theta}(k+1\,|\,k) + K(k+1)y(k+1) \tag{58}$$

To derive the covariance update equation, we subtract $\theta(k+1)$ from both sides of Eq. (58) and use Eqs. (48) and (51) to obtain

$$\tilde{\theta}(k+1\,|\,k+1) = [I - K(k+1)H(k+1)]\tilde{\theta}(k+1\,|\,k) - K(k+1)n(k+1) \tag{59}$$

from which the familiar, Joseph-form, covariance update equation results

$$P(k+1\,|\,k+1) = [I - K(k+1)H(k+1)]P(k+1\,|\,k) \times [I - K(k+1)H(k+1)]^T + K(k+1)R(k+1)K^T(k+1) \tag{60}$$

where $P(k+1\,|\,k+1) \triangleq E\{\tilde{\theta}(k+1\,|\,k+1)\tilde{\theta}^T(k+1\,|\,k+1)\}$ is the filtering error covariance matrix.

*Remark 6.* In practice, numerically stable square-root algorithms[19] should be preferred to using the conventional covariance update, Eq. (60).

*Attitude Matrix Update*

To compute the measurement-updated attitude matrix at time $t_{k+1}$, we use the most recent estimate of the parameter vector $\hat{\theta}(k+1\,|\,k+1)$, the estimated attitude matrix corresponding to time $t_k$, and the measured angular velocity matrix, defined as

$$\hat{\Omega}(k) \triangleq -[\hat{\omega}(k)\times] \tag{61}$$

in Eq. (17). This yields

$$\hat{D}(k+1\,|\,k+1) = \Big\{I + \hat{A}(k+1,k) + \tfrac{1}{2}\hat{A}^2(k+1,k) + \tfrac{1}{6}\hat{A}^3(k+1,k) + \tfrac{1}{6}T[\hat{A}(k+1,k)\hat{\Omega}(k) - \hat{\Omega}(k)\hat{A}(k+1,k)]\Big\}\hat{D}^*(k\,|\,k) \tag{62}$$

where the a posteriori estimate of $A(k+1,k)$ is defined as

$$\hat{A}(k+1,k) \triangleq -[\hat{\theta}(k+1\,|\,k+1)\times] \tag{63}$$

and $\hat{D}^*(k\,|\,k)$ is the a posteriori, orthogonalized estimate of the attitude matrix at time $t_k$, to be discussed in the sequel.

*Remark 7.* Equation (62) is based on a third-order approximation of the attitude matrix using the updated estimates of the IRP vector. Obviously, the accuracy of this third-order approximation relies on the assumption that these parameters are small. In fact, it will be shown in the sequel that this is always the case, because the updated parameters are reset to zero after each measurement update. Moreover, the components of the parameter vector at each data point can always be kept small by selecting a sufficiently small discretization interval.

*Estimate Reset*

As shown in Eq. (62), the a posteriori attitude matrix, $\hat{D}(k+1\,|\,k+1)$, is computed based on the a posteriori estimate $\hat{\theta}(k+1\,|\,k+1)$. This estimate of the attitude matrix is then used in consecutive prediction and filtering steps, which in turn implies a full reset control[20] of the parameter vector

$$\theta^c(k+1) = \theta(k+1) - \hat{\theta}(k+1\,|\,k+1) \tag{64}$$

where $\theta^c(k+1)$ is the reset state vector at $t_{k+1}$, and a corresponding reset of the state estimate

$$\hat{\theta}^c(k+1\,|\,k+1) = 0 \tag{65}$$

which is then used in the ensuing time propagation step.

*Remark 8.* Notice that because the reset control is applied to both the state vector and its estimate, no changes are necessary in the estimation error covariance matrix.

**Attitude Matrix Orthogonalization**

Although the true attitude matrix is orthogonal, the filtered DCM will not be orthogonal, due to numerical implementation errors and the approximate nature of the third-order formula used to compute the attitude from the estimated parameters. To improve the algorithm's accuracy and to enhance its stability, an additional orthogonalization stage is introduced into the estimator, immediately following the measurement update stage. In the orthogonalization stage, the filtered attitude matrix is orthogonalized, that is, the orthogonal matrix closest to the filtered attitude matrix is found. This orthogonal matrix is then propagated to the next measurement update point.

In the sequel, the Euclidean norm (2-norm) will be used for vectors, and the Frobenius norm (F-norm) will be used for matrices.

Given the a posteriori attitude matrix $\hat{D}(k+1\,|\,k+1)$, the matrix orthogonalization problem is to find

$$\hat{D}^*(k+1\,|\,k+1) \triangleq \arg\min_C \|\hat{D}(k+1\,|\,k+1) - C\| \tag{66}$$

subject to

$$C^T C = I \tag{67}$$

Being a special case of the orthogonal Procrustes problem,[21] the matrix orthogonalization problem can be easily solved using the singular value decomposition (SVD). Thus, if

$$\hat{D}(k+1\,|\,k+1) = U(k+1)\Sigma(k+1)V^T(k+1) \tag{68}$$

is the SVD of the matrix $\hat{D}(k+1\,|\,k+1)$ where $U(k+1)$ and $V(k+1)$ are the left and right singular vector matrices, respectively, and $\Sigma(k+1)$ is the singular value matrix, then

$$\hat{D}^*(k+1\,|\,k+1) = U(k+1)V^T(k+1) \tag{69}$$

The excessive computational burden associated with the SVD might render its use prohibitive in certain applications, e.g., in real-time attitude determination and control. In such cases, an alternative orthogonalization scheme, introduced by Bar-Itzhack and Meyer,[22] can be used. According to this scheme, the orthogonalized matrix $\hat{D}^*(k+1\,|\,k+1)$ can be computed iteratively using the recursion

$$X_{j+1} = \tfrac{3}{2}X_j - \tfrac{1}{2}X_j X_j^T X_j, \qquad X_0 = \hat{D}(k+1\,|\,k+1) \tag{70}$$

where $X_j \xrightarrow{j\to\infty} \hat{D}^*(k+1\,|\,k+1)$. This scheme was shown in Ref. 22 to be globally convergent and to possess a quadratic convergence rate.

Noting the fast convergence rate of the recursive orthogonalization method just shown, this scheme is incorporated into our estimation algorithm using just a single step of the recursion (70). Thus, an improved (nearly orthogonal), a posteriori estimate for the attitude matrix is computed as

$$\hat{D}^*(k+1\,|\,k+1) = N(k+1)\hat{D}(k+1\,|\,k+1) \tag{71}$$

where the linear transformation that maps the a posteriori attitude matrix into its orthogonal version is defined by

$$N(k+1) \triangleq \tfrac{3}{2}I - \tfrac{1}{2}\hat{D}(k+1\,|\,k+1)\hat{D}^T(k+1\,|\,k+1) \tag{72}$$

**DCM Orthogonalization: Analysis**

The introduction of the external orthogonalization step into the estimator may conceivably affect its performance and statistical characteristics, thus calling for appropriate adjustments in the algorithm to preserve its theoretical properties. In the remainder of this section, the possible effects of the orthogonalization procedure (71) on the a posteriori state estimate and error covariance matrix are analyzed. In fact, as the next theorem shows, to first-order accuracy the orthogonalization procedure does not affect the estimator.

*Theorem 1.* Let $\hat{D}^*(k+1\,|\,k+1)$ denote the orthogonalized version of $\hat{D}(k+1\,|\,k+1)$, computed in Eq. (71). Then

$$\hat{D}^*(k+1\,|\,k+1) = [I + \eta[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)]]\hat{D}(k+1\,|\,k+1) \tag{73}$$

where $\eta[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)]$ is a matrix-valued function that satisfies

$$\|\eta[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)]\| \sim \mathcal{O}(\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2) \tag{74}$$

*Proof.* To prove the theorem we need to show that

$$N(k+1) = I + \eta[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)] \tag{75}$$

with $\eta[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)]$ satisfying Eq. (74). To this end, rewrite Eq. (62) as

$$\hat{D}(k+1\,|\,k+1) = \Phi[\hat{A}(k+1,k)]\hat{D}^*(k\,|\,k) \tag{76}$$

which is an implied definition of the matrix-valued function $\Phi()$. Using Eq. (76) in Eq. (72) and noting the orthogonality of $\hat{D}^*(k\,|\,k)$ yields

$$N(k+1) = \tfrac{3}{2}I - \tfrac{1}{2}\Phi[\hat{A}(k+1,k)]\Phi^T[\hat{A}(k+1,k)] \tag{77}$$

Using Eq. (63) we have

$$\hat{A}^2(k+1,k) = -\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2 I + \hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\hat{\boldsymbol{\theta}}^T(k+1\,|\,k+1) \tag{78a}$$

$$\hat{A}^3(k+1,k) = \|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times] \tag{78b}$$

whence

$$\Phi[\hat{A}(k+1,k)] = I - [\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times] + \mu_1 + \mu_2 + \mu_3 \tag{79}$$

where the following definitions have been used:

$$\mu_1 \triangleq \tfrac{1}{2}[-\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2 I + \hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\hat{\boldsymbol{\theta}}^T(k+1\,|\,k+1)] \tag{80a}$$

$$\mu_2 \triangleq \tfrac{1}{6}\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2[\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times] \tag{80b}$$

$$\mu_3 \triangleq \tfrac{1}{6}[(\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times\hat{\boldsymbol{\omega}}(k)T)\times] \tag{80c}$$

It is easy to show that

$$\|\mu_1\| \leq \sqrt{\tfrac{3}{2}}\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2 \tag{81}$$

$$\|\mu_2\| = \frac{1}{3\sqrt{2}}\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^3 \tag{82}$$

Consider now the vector product $\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times\hat{\boldsymbol{\omega}}(k)T$, appearing in Eq. (80c). As will be shown in the sequel [see the state prediction equation (88)], regarding $\hat{\boldsymbol{\omega}}(t)$ as approximately constant over the small sampling interval $[t_k, t_{k+1}]$ yields

$$\hat{\boldsymbol{\omega}}(k)T \approx \hat{\boldsymbol{\theta}}(k+1\,|\,k) \tag{83}$$

Hence, using Eq. (57) we have

$$\begin{aligned}\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times\hat{\boldsymbol{\omega}}(k)T\| &= \|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\times\widehat{\delta\boldsymbol{\theta}}(k+1\,|\,k+1)\| \\ &\leq \|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|\|\widehat{\delta\boldsymbol{\theta}}(k+1\,|\,k+1)\| \\ &\leq \|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2\end{aligned} \tag{84}$$

Using Eq. (84) in Eq. (80c) yields

$$\|\mu_3\| \leq \frac{1}{3\sqrt{2}}\|\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)\|^2 \tag{85}$$

Using Eqs. (81), (82), and (85) in Eq. (79) and substituting the result in Eq. (77) then yields Eq. (75), completing the proof. □

Now, from Eq. (73) we conclude that, after the initial transient period, the effects of the orthogonalization procedure on the filtered DCM are only of second order in $\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)$. Hence, to first-order accuracy, no changes in $\hat{\boldsymbol{\theta}}(k+1\,|\,k+1)$ are necessary as a result of the orthogonalization, and consequently no changes are required in the a posteriori covariance matrix.

### Prediction

In the prediction step, the reset a posteriori state estimate at time $t_k$ and its corresponding error covariance matrix, $\hat{\boldsymbol{\theta}}^c(k\,|\,k)$ and $P(k\,|\,k)$, respectively, are propagated to time $t_{k+1}$.

Noting Eqs. (15) and (64) we have, after the state reset at time $t_k$,

$$\boldsymbol{\theta}(k+1) = \boldsymbol{\theta}^c(k) + \int_{t_k}^{t_{k+1}}\boldsymbol{\omega}(\tau)\,\mathrm{d}\tau = \tilde{\boldsymbol{\theta}}(k\,|\,k) + \int_{t_k}^{t_{k+1}}\boldsymbol{\omega}(\tau)\,\mathrm{d}\tau \tag{86}$$

Hence, because the estimator is unbiased, the predicted state at $t_{k+1}$ is

$$\hat{\boldsymbol{\theta}}(k+1\,|\,k) = \int_{t_k}^{t_{k+1}}\boldsymbol{\omega}(\tau)\,\mathrm{d}\tau \tag{87}$$

In practice, however, we only have access to the measured value of the angular velocity. Thus, using the measured velocity in Eq. (87) yields the following state prediction equation:

$$\hat{\boldsymbol{\theta}}(k+1\,|\,k) = \int_{t_k}^{t_{k+1}}\hat{\boldsymbol{\omega}}(\tau)\,\mathrm{d}\tau \tag{88}$$

Subtracting Eq. (88) from Eq. (86) and noting Eq. (18), the corresponding prediction error equation is

$$\tilde{\boldsymbol{\theta}}(k+1\,|\,k) = \tilde{\boldsymbol{\theta}}(k\,|\,k) - \int_{t_k}^{t_{k+1}}\delta\boldsymbol{\omega}(\tau)\,\mathrm{d}\tau \tag{89}$$

Noting that the two terms in the RHS of Eq. (89) are uncorrelated, the following, trivially simple covariance propagation equation results:

$$P(k+1\,|\,k) = P(k\,|\,k) + \int_{t_k}^{t_{k+1}}Q(\tau)\,\mathrm{d}\tau \tag{90}$$

*Remark 9.* Any quadrature formula can be used in Eq. (88). Simpson's quadrature scheme yields

$$\hat{\boldsymbol{\theta}}(k+1\,|\,k) = \tfrac{1}{6}T\left[\hat{\boldsymbol{\omega}}(k) + 4\hat{\boldsymbol{\omega}}\left(k+\tfrac{1}{2}\right) + \hat{\boldsymbol{\omega}}(k+1)\right] \tag{91}$$

where $\hat{\boldsymbol{\omega}}(k+\frac{1}{2}) \triangleq \hat{\boldsymbol{\omega}}[t_k + (T/2)]$. The selection of quadrature scheme should be based mainly on the expected spacecraft dynamics; thus, for a slow-dynamics case, simple trapezoidal integration will probably suffice. Notice also that in many spacecraft applications, rate-integrating gyros are used, which give $\hat{\boldsymbol{\theta}}(k+1\,|\,k)$ directly (rendering the prediction stage even simpler).

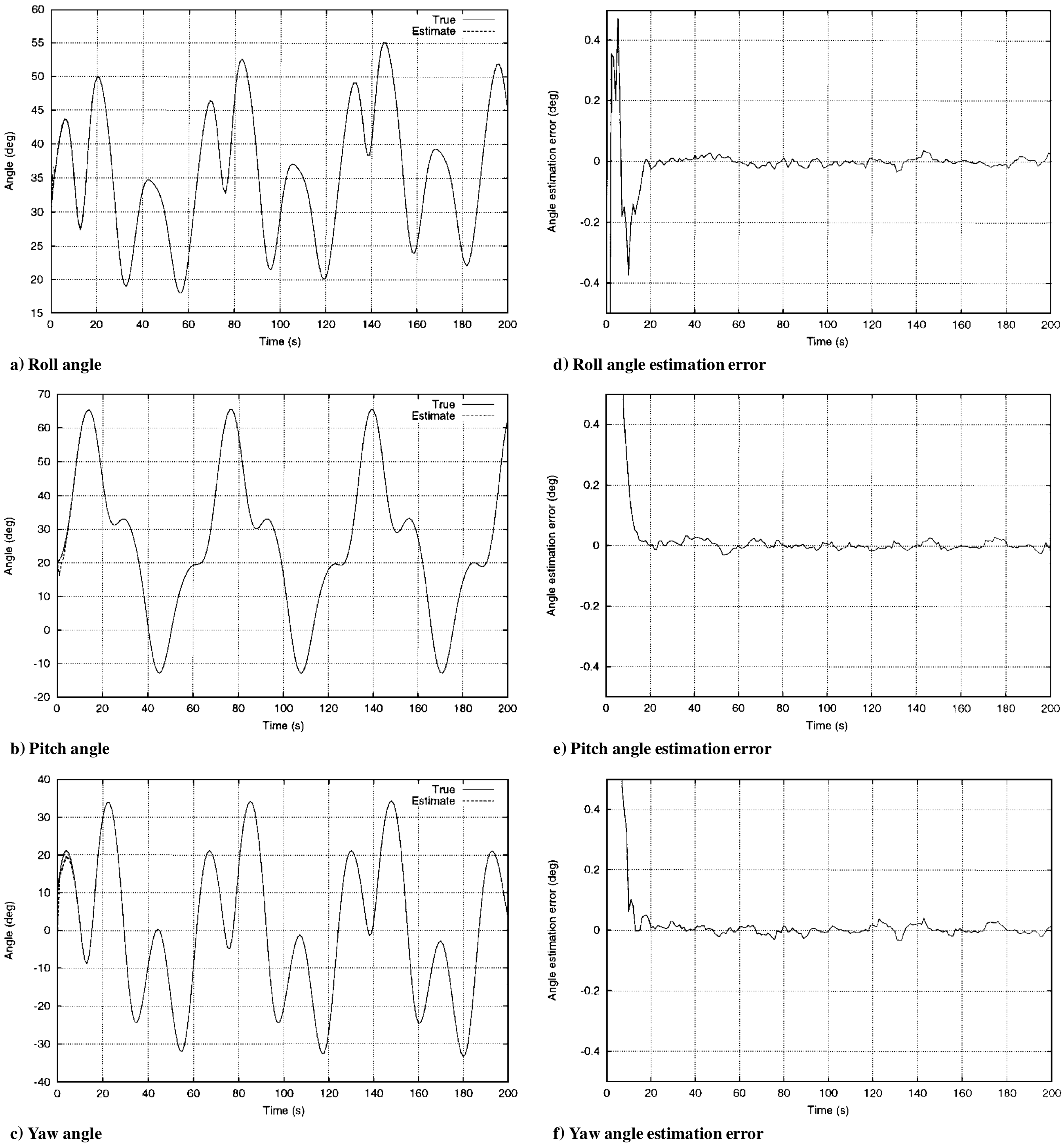
*Remark 10.* If the intensity of the RG measurement noise can be assumed to be time invariant (as is often the case), then the error covariance propagation is simply

$$P(k+1\,|\,k) = P(k\,|\,k) + QT \tag{92}$$

If, however, $Q(t)$ is time varying, Simpson's integration (or some other quadrature formula) can be used.

### Attitude Matrix Prediction

To predict the attitude matrix at $t_{k+1}$ we use the most recent estimate of the parameter vector $\hat{\boldsymbol{\theta}}(k+1\,|\,k)$, the orthogonalized




a) Roll angle

b) Pitch angle

c) Yaw angle

d) Roll angle estimation error

e) Pitch angle estimation error

f) Yaw angle estimation error

**Fig. 1 Euler angle and estimation error time histories.**

estimate of the attitude matrix corresponding to $t_k$, and the measured angular velocity matrix, in Eq. (17). This yields

$$\hat{D}(k+1\,|\,k) = \Big\{I + \bar{A}(k+1,k) + \tfrac{1}{2}\bar{A}^2(k+1,k) + \tfrac{1}{6}\bar{A}^3(k+1,k) + \tfrac{1}{6}T[\bar{A}(k+1,k)\hat{\Omega}(k) - \hat{\Omega}(k)\bar{A}(k+1,k)]\Big\}\hat{D}^*(k\,|\,k) \tag{93}$$

where the a priori estimate of $A(k+1,k)$ is defined as

$$\bar{A}(k+1,k) \triangleq -[\hat{\boldsymbol{\theta}}(k+1\,|\,k)\times] \tag{94}$$

## Numerical Example

To demonstrate the performance of the new attitude estimation algorithm, a numerical simulation study was conducted, in which simulated vector measurements and RG data were processed by the new estimator to obtain the estimated attitude matrix at each measurement processing point.

The standard deviation of the gyro noise power spectral density was 0.01 deg/h$^{1/2}$. Both the body-frame and the reference frame vector measurements were contaminated by zero-mean, white, Gaussian noise sequences, orthogonal to the true directions, which were generated via the following algorithms:

$$\boldsymbol{n}_u(k+1) = x_u(k+1)\frac{\boldsymbol{w}_u(k+1)\times\boldsymbol{u}_0(k+1)}{\|\boldsymbol{w}_u(k+1)\times\boldsymbol{u}_0(k+1)\|} \tag{95}$$

$$\boldsymbol{n}_v(k+1) = x_v(k+1)\frac{\boldsymbol{w}_v(k+1)\times\boldsymbol{v}_0(k+1)}{\|\boldsymbol{w}_v(k+1)\times\boldsymbol{v}_0(k+1)\|} \tag{96}$$

where $\boldsymbol{w}_u(k+1)$ and $\boldsymbol{w}_v(k+1)$ are randomly chosen vectors and $x_u(k+1)$ and $x_v(k+1)$ are normal deviates satisfying

$$x_u(k+1) \sim \mathcal{N}\left(0, \sigma_u^2\right), \qquad x_v(k+1) \sim \mathcal{N}\left(0, \sigma_v^2\right) \tag{97}$$

implying

$$R_v = \sigma_v^2 I, \qquad R_u = \sigma_u^2 I \tag{98}$$

The noise equivalent angles were set to $\sigma_u = \sigma_v = 100$ arc-s. Notice that the values assumed for both the gyro white noise drift and the star tracker noise are very conservative, compared to the current technology state of the art.[3,23]

The initial attitude estimate was set to the identity matrix (thus assuming that $\mathcal{S}_u$ and $\mathcal{S}_v$ coincide at $t_0$) whereas the true attitude corresponded to Euler angles of 30, 20, and 10 deg in roll, pitch, and yaw, respectively. Again, this constitutes a conservative assumption, as we can always use the first vector measurements to find a rough initial estimate of the attitude using some point-estimation scheme, e.g., QUEST,[24] or the approximate initialization method suggested in Ref. 13. However, it was found that there was no need to use such an initialization scheme, as all simulation runs starting at the identity matrix successfully converged (a detailed Monte Carlo study of the convergence of the algorithm and the orthogonality of the estimated attitude matrix appears in Ref. 25).

The angular velocity of $\mathcal{S}_v$ relative to $\mathcal{S}_u$ was chosen to be

$$\boldsymbol{\omega}(t) = \begin{bmatrix} 2\sin(0.2t + \pi/4) \\ 3\sin(0.1t + \pi/2) \\ 6\sin(0.3t + 3\pi/4) \end{bmatrix} \text{deg/s} \tag{99}$$

i.e., an angular velocity with time-varying direction. The filter was run at a rate of 20 Hz, i.e., the sampling interval was $T = 0.05$ s, whereas the measurement processing rate was a slow 1 Hz.

Figure 1 presents the three true Euler angles, the estimated angles (computed using the estimated attitude matrix) and the corresponding estimation errors, in a typical run. The Euler angle sequence assumed was 3–2–1. The steady-state estimation errors of the Euler angles computed using the estimated DCM in a typical run were smaller than 0.015 deg (1-$\sigma$). As can be clearly evidenced from Fig. 1, the estimator's performance was not affected by the changing direction of the angular velocity vector, thus demonstrating the robustness of the new algorithm.

## Conclusions

A computationally efficient, nonlinear estimation algorithm has been presented that uses vector measurements and gyro readings to estimate the DCM. The algorithm is based on a recently introduced, third-order minimal parameterization of the attitude matrix. This facilitates the use of a three-dimensional filter to estimate the nine-parameter attitude matrix.

The extremely simple kinematics obeyed by the particular parameter vector chosen is inherited by the estimator's time propagation equations, which results in the filter's high numerical efficiency. The DCM orthogonality constraint is dealt with by incorporating an orthogonalization procedure following the measurement update stage. Based on a single-step implementation of an iterative orthogonalization technique, the incorporation of this procedure into the estimator was shown to not require any further modifications in the structure of the algorithm.

A numerical simulation study that demonstrates the performance of the proposed algorithm has been presented. Assuming conservative gyro noise levels and vector observation errors, the attitude estimated via the new algorithm has been shown to be accurate and robust with respect to initialization errors in a case with a direction-changing angular velocity.

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