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ACTIVE DISTRIBUTED VIBRATION CONTROL OF ANISOTROPIC PIEZOELECTRIC LAMINATED PLATES

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A design strategy is proposed for the active vibration control of fully anisotropic plates in which the active elements are laminated, spatially distributed, piezoelectric layers. The control methodology results from stability criteria established through the second method of Lyapunov, and is based on the consideration of the total system energy. The results show that for a fully anisotropic plate it is sufficient to ensure asymptotic stability provided that three criteria are met: (1) for each piezoelectric actuator laminate above the composite structure mid-plane there exists a corresponding identically polarized sensor laminate, also located above the mid-plane; (2) a linear control law governing each conjugate sensor/actuator pair is enforced such that the input to a given actuator is always proportional and opposite in sign to the current induced by the corresponding sensor; (3) for each conjugate pair above the mid-plane there exists an identical pair below the mid-plane. The analysis shows that these design prerequisites may be relaxed significantly in the absence of bending–stretching coupling. When the design criteria are satisfied, the measure of active vibration suppression becomes directly dependent on the choice of transducer spatial distribution functions. A weaker sufficient criterion is also identified, which could potentially be utilized to relax the design constraints further for fully anisotropic media. Previously employed design strategies for bending vibration control of beams and isotropic plates are shown to be a subclass of general anisotropic plate control theory, and often destabilizing in the presence of anisotropy.

1. INTRODUCTION

The component elements used in space systems are generally flexible, lightly damped, and have a large number of vibrational modes. Mission requirements often preclude the use of passive damping treatments. Active dampers traditionally have been based on the implementation of a finite number of discrete transducers [1, 2]. Because the component members are continuous and in theory possess an infinite number of degrees of freedom, these control schemes truncate the system model into a finite number of discrete modes. Such methodologies necessarily lead to the spillover of unmodelled high frequency modes, which tends to reduce controller effectiveness and robustness and may lead to instability [2].

Within the past decade vibration control techniques which utilize distributed actuators and sensors have been developed. Spatially distributed actuators were shown to provide successfully active damping to beams without requiring discretization of the system model [3–7]. These studies indicate that active vibration control of Bernoulli–Euler beams with nearly arbitrary boundary constraints may be accomplished by using distributed actuators with strain fields which vary spatially. The distributed actuators were designed by using polyvinylidine fluoride (PVF2), a polymeric material made piezoelectrically active through a polarization process which is applied during its manufacture [8]. Burke and Hubbard

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showed that spatial variation of the actuator polarization profile allows for the simultaneous control of all modes or the selective control of desired modal subsets [4]. A reciprocal sensor model was developed by Miller and Hubbard [9] and then incorporated into a component system in which distributed sensor and actuator transducers were integrated into a component beam member, yielding *smart* structural members that could be used to form complex structures [10, 11]. In their analysis, stability criteria were established which govern both the design of piezoelectrically laminated Bernoulli-type beam components and their integration into larger complex superstructures.

Since many aerospace structures contain flexible components which are characteristically two-dimensional, much effort has been recently focused on extending these one-dimensional beam formulations to the control of plates. Burke and Hubbard [12] developed a formulation for the control of thin elastic (Kirchhoff–Love) isotropic plates subject to nearly general boundary conditions, in which the active elements were spatially varying biaxially polarized piezoelectric transducer layers. Stability criteria were established and the effects of spatial weighting were discussed. Lee [13, 14] generalized the classical laminate theory developed by Ashton *et al*. [15] to include the effect of laminated piezoelectric layers. The theory exploits the piezoelectric phenomenon to provide for the distributed actuation and sensing of bending, torsion, shearing, shrinking and stretching in flexible anisotropic plates.

The asymmetric behavior of anisotropic laminated piezoelectric plates forces one to ascertain what criteria must be established in order to guarantee the active dissipation of vibrational energy in the distributed control problem involving such a structure. Applications of the direct velocity feedback laws typified in the control of isotropic beams [5, 10] and plates [12] are not necessarily extensible to anisotropic structures. Therefore, in this paper, stability criteria and an active vibration suppression control design methodology for anisotropic laminated piezoelectric plates are derived. The criteria that are established are sufficient for asymptotic stability based on the second (direct) method of Lyapunov. The general equations of motion as developed by Lee [13] are incorporated into a Lyapunov functional which is representative of the total system mechanical energy. The boundary conditions as developed in Reference [13] are then generalized to include dissipative loss mechanisms and applied to the functional, leading to the following three easily realizable design constraints which are sufficient to guarantee the asymptotic stability of a piezoelectrically active laminated anisotropic rectangular plate: (1) for each piezoelectric actuator laminate above the composite structure mid-plane there must exist a corresponding identically polarized sensor laminate, located above the mid-plane; (2) a linear control law governing each conjugate sensor/actuator pair is enforced such that the input to a given actuator is always proportional and opposite in sign to the current induced by the corresponding sensor; (3) for each conjugate pair above the mid-plane there exists an identical pair below the mid-plane. The analysis shows that these design criteria may be significantly relaxed in the absence of coupling between stretching and bending. When these constraints are satisfied, then active vibration suppression becomes directly dependent on the choice of transducer spatial distribution functions. Weaker criteria, sufficient to ensure asymptotic stability, are also established which may be exploited in certain instances to further relax the design constraints. Previously employed design strategies for bending vibration control of beams [10, 11] and isotropic plates [12] are shown to be potentially destabilizing in the presence of anisotropy.

2. DEVELOPMENT OF STABILITY CRITERIA

The geometry of the general system under consideration is described in Figure 1. For simplicity, and without loss of generality, a rectangular anisotropic plate with n piezoelec-

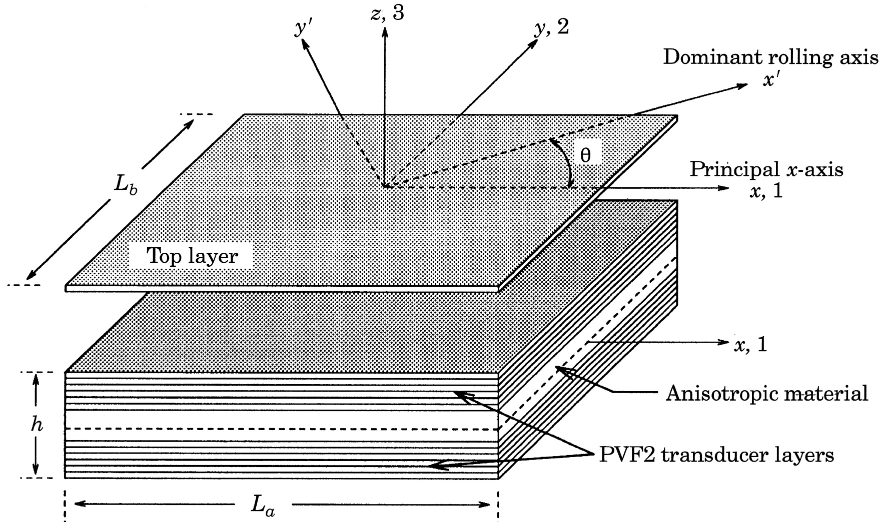


Figure 1. The geometry of the general piezoelectric laminate system.

trically active laminate layers stacked upon each face is to be considered. As shown in the figure, the length dimensions of the system in the x and y directions are denoted as L_a and L_b, respectively. Exactly n layers function as actuators and n corresponding layers function as sensors. Actuator and sensor layers may be located both above and below the neutral mid-plane. Each piezoelectric layer may be independently anisotropic, although typically the active layers are either transversely isotropic, or else their mechanical stiffness relative to the substrate allows their anisotropy to be neglected. The polarization of each laminate electric field may be spatially varied through varying the surface electrode pattern of each layer. The thickness of each layer is assumed to be constant throughout the dominant plane of the plate. The poling direction of each layer is assumed to be outwardly normal with respect to the mid-plane of the system. The strain–displacement relationships for each laminate are assumed to be governed by the Kirchhoff–Love approximation, in which displacements of the laminae are related to each other linearly through the thickness direction. Finally, the dominant poling axis, x_k', of each piezoelectric laminate may be rotated from the principal x-axis through a skew angle, θ, defined in a positive right-hand sense about the z-axis as illustrated in Figure 1. Lee [13] has shown that the introduction of such rotations induces a torsional effect in piezoelectric actuators which possess transverse isotropy in the x–y plane (and correspondingly leads to the detection of shear strain in compatible sensors).

The development of general stability criteria and subsequently a control design strategy for general rectangular anisotropic piezoelectric laminate systems typified in Figure 1 emerge through choosing a Lyapunov functional which is representative of the total kinetic and strain energy inherent in the system. If the time derivative of a Lyapunov functional is negative definite, then the system is asymptotically stable. Before introducing the energy-based functional, the equations of motion as set forth by Lee [13] are first established in a form most suitable to the forthcoming analysis. The boundary conditions as developed by Lee are then generalized to include piezoelectric effects and lumped dissipative elements. The energy-based Lyapunov functional is then introduced and the system energy flux is established. The equations of motion and boundary conditions are then folded into the energy flux expression, yielding criteria sufficient to ensure asymptotic stability and, subsequently, a control design methodology.

2.1. EQUATIONS OF MOTION

Given that u^0 and v^0 are the displacements of the system mid-plane in the x and y directions respectively, the equations of motion as developed in reference [13] for the general system described in Figure 1 may be written as

$$\rho h u^0_{tt} = \mathscr{L}_{s1} - \mathscr{L}_{c1} - \sum_{k=1}^{n} e^{0k}_{31} V^k_x - \sum_{k=1}^{n} e^{0k}_{36} V^k_y, \tag{1}$$

$$\rho h v^0_{tt} = \mathscr{L}_{s2} - \mathscr{L}_{c2} - \sum_{k=1}^{n} e^{0k}_{36} V^k_x - \sum_{k=1}^{n} e^{0k}_{32} V^k_y, \tag{2}$$

$$\rho h w_{tt} = -\mathscr{L}_b + \mathscr{L}_{c3} + \mathscr{L}_{c4} - \sum_{k=1}^{n} e^{0k}_{31} z^k_0 V^k_{xx} - 2\sum_{k=1}^{n} e^{0k}_{36} z^k_0 V^k_{xy} - \sum_{k=1}^{n} e^{0k}_{32} z^k_0 V^k_{yy}, \tag{3}$$

where the expressions $\mathscr{L}_{s1}$, $\mathscr{L}_{s2}$, $\mathscr{L}_{c1}$, $\mathscr{L}_{c2}$, $\mathscr{L}_{c3}$, $\mathscr{L}_{c4}$ and $\mathscr{L}_b$ are defined as

$$\mathscr{L}_{s1} = A_{11}u^0_{xx} + 2A_{16}u^0_{xy} + A_{66}u^0_{yy} + A_{16}v^0_{xx} + (A_{12} + A_{66})v^0_{xy} + A_{26}v^0_{yy}, \tag{4}$$

$$\mathscr{L}_{s2} = A_{16}u^0_{xx} + (A_{12} + A_{66})u^0_{xy} + A_{26}u^0_{yy} + A_{66}v^0_{xx} + 2A_{26}v^0_{xy} + A_{22}v^0_{yy}, \tag{5}$$

$$\mathscr{L}_{c1} = B_{11}w_{xxx} + 3B_{16}w_{xxy} + (B_{12} + 2B_{66})w_{xyy} + B_{26}w_{yyy}, \tag{6}$$

$$\mathscr{L}_{c2} = B_{16}w_{xxx} + (B_{12} + 2B_{66})w_{xxy} + 3B_{26}w_{xyy} + B_{22}w_{yyy}, \tag{7}$$

$$\mathscr{L}_{c3} = B_{11}u^0_{xxx} + 3B_{16}u^0_{xxy} + (B_{12} + 2B_{66})u^0_{xyy} + B_{26}u^0_{yyy}, \tag{8}$$

$$\mathscr{L}_{c4} = B_{16}v^0_{xxx} + (B_{12} + 2B_{66})v^0_{xxy} + 3B_{26}v^0_{xyy} + B_{22}v^0_{yyy}, \tag{9}$$

$$\mathscr{L}_b = D_{11}w_{xxxx} + 4D_{16}w_{xxxy} + 2(D_{12} + 2D_{66})w_{xxyy} + 4D_{26}w_{xyyy} + D_{22}w_{yyyy}. \tag{10}$$

In these expressions, A_{mn}, B_{mn} and D_{mn} ($m, n = 1, 2, 6$) are the laminated plate constitutive constants and the subscripts indicate partial differentiation. The piezoelectric stress-charge constants of each lamina with respect to the laminate axes, e^0_{31}, e^0_{32}, and e^0_{36}, are direct functions of the skew angle, θ, as defined in Figure 1. The subscript notation is consistent with IEEE compact matrix notation [16], and thus the presence of the e^0_{36} term infers the induction and detection of shear forces, whereas e^0_{31} and e^0_{32} infer the induction and detection of transverse bending moments and angular deflections. The plate constitutive constants are equivalent moduli which are representative of the conglomerate laminated structure in which the mechanical properties of the sensor, actuator and substrate layers are implicitly recognized. The summations in equations (1)–(3) pertain uniquely to the n actuator laminae to which the control voltages $V^k(x, y, t)$ are applied. The control voltages vary both spatially and temporally. The equivalent density, ρ, is defined as

$$\rho = \sum_{k=-n}^{n} \frac{\rho_k h_k}{h}, \tag{11}$$

where ρ_k represents the density of the kth lamina, h_k is the thickness of the kth lamina, and h is the thickness of the composite structure. In the preceding equation, $k < 0$ refers to the sensor layers, $k = 0$ refers to the anisotropic plate substrate, and $k > 0$ refers to the actuator layers. The z co-ordinate of the mid-plane of the kth layer, z^k_0, is defined as

$$z^k_0 = \tfrac{1}{2}(z_k + z_{k-1}). \tag{12}$$

Note that the expressions $\mathscr{L}_{s1}$ and $\mathscr{L}_{s2}$ contain only those terms relative to membrane stretching, the $\mathscr{L}_b$ expression contains terms unique to transverse bending and twisting, and the remaining expressions $\mathscr{L}_{c1}$ through $\mathscr{L}_{c4}$ are related to the coupling between stretching, bending and twisting. The full anisotropic case therefore degenerates into orthotropy (general mid-plane symmetric laminates) if $\mathscr{L}_{c1}$ through $\mathscr{L}_{c4}$ are identically zero. The specially orthotropic and isotropic cases are derived by setting certain constitutive constants to zero while establishing others in accord with the well-known moduli, as put forward in the literature [15, 17, 18].

2.2. BOUNDARY CONDITIONS

The boundary conditions identified in reference [13], which directly pertain to the anisotropic piezo-laminated plate system the equations of motion of which are described by equations (1)–(3), fail to include dissipative loss mechanisms which arise due to non-ideal constraints. Inclusion of these loss mechanisms, while leading to a more accurate mathematical description of the real system, will ultimately lead to a control methodology which will be more generally applicable. For the sake of simplicity, instead of applying Hamilton's principle directly to the anisotropic plate problem in order to derive a more general expression for the boundary constraints, the identical result is obtained through the direct consideration of an equivalent *isotropic* system. Whereas Hamilton's principle will lead to both equations of motion and boundary conditions for the isotropic composite plate, only the boundary conditions will be of direct interest. By rewriting the boundary conditions for the isotropic system in terms of force and moment resultants rather than displacements, the identical boundary conditions as determined in reference [13] for *anisotropic* laminated plates are acquired, with the exception that they include dissipative loss mechanisms as well.

The expanded derivation of boundary conditions in which loss mechanisms due to non-ideal constraints are considered is based on a laminated plate system the geometry of which is given in Figure 1, but where the composite structure is assumed to behave as a fully isotropic (rather than anisotropic) material. The boundary conditions will be established without a loss of generalization so as to apply to fully anisotropic plates as well, and as such will be completely compatible with (but more general than) those given in reference [13]. The boundary conditions will be presented in a form compatible with Poisson's derivation [18]. Kirchhoff proved that the boundary conditions as developed by Poisson for a homogeneous plate are too numerous [18]. Even though the Kirchhoff equivalent shear force boundary conditions are equally applicable to piezoelectric laminates [13], nonetheless the Poisson form will prove the most convenient and advantageous for determinig stability criteria.

From Hamilton's principle it is known that a geometrically admissible motion of the system between prescribed configurations at arbitrary times t_1 and t_2 satifies the dynamic force requirements if and only if [19]

$$\int_{t_1}^{t_2} [\delta(T - \mathscr{U}) + \delta W]\, \mathrm{d}t = 0, \tag{13}$$

where the symbol δ indicates geometric variation. The left side of equation (13) is commonly referred to as the variational indicator [19]. T and $\mathscr{U}$ are defined respectively as the kinetic and potential energies, while W represents the work acting on the system due to external forces. In the following analysis, the virtual work expression will include both dissipative and piezoelectric forces.

2.2.1. *Virtual work due to piezoelectric forces*

The active piezoelectric laminate layers induce moments and forces on the structure and thus contribute to expressions of virtual work in the description of the total system energy. These moments and forces are quasistatically equivalent to the stress system depicted in Figure 2, in which equivalent stress and moment resultants are applied at the mid-plane.

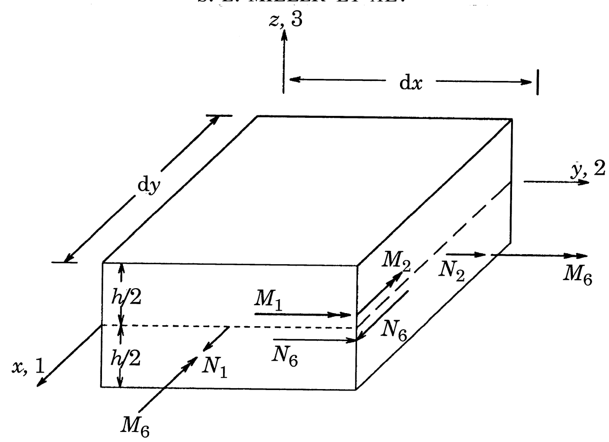


Figure 2. The equivalent stress state at the mid-plane.

The total moment resultants acting on the structure are quantified as bending moments per unit length and are defined as

$$(M_1, M_2, M_6) = \sum_{k=1}^{n} \int_{-h/2}^{h/2} ((\sigma^x)^k, (\sigma^y)^k, (\sigma^{xy})^k) z \, dz, \tag{14}$$

where $(\sigma^x)^k$, $(\sigma^y)^k$ and $(\sigma^{xy})^k$ are the normal and in-plane shear stresses of the kth layer. The total normal force resultants acting on the structure are measured in terms of forces per unit length and are defined as

$$(N_1, N_2, N_6) = \sum_{k=1}^{n} \int_{-h/2}^{h/2} ((\sigma^x)^k, (\sigma^y)^k, (\sigma^{xy})^k) \, dz. \tag{15}$$

The contribution to these moment and force resultants due exclusively to the piezoelectric effect of the active laminae have been previously determined [13, 20]. The piezoelectrically induced moment resultants are given by

$$\begin{bmatrix} M_1^P \\ M_2^P \\ M_6^P \end{bmatrix} = \sum_{k=1}^{n} z_0^k V^k \begin{bmatrix} e_{31}^0 \\ e_{32}^0 \\ e_{36}^0 \end{bmatrix}^k, \tag{16}$$

and the piezoelectrically induced force resultants are given as

$$\begin{bmatrix} N_1^P \\ N_2^P \\ N_6^P \end{bmatrix} = \sum_{k=1}^{n} V^k \begin{bmatrix} e_{31}^0 \\ e_{32}^0 \\ e_{36}^0 \end{bmatrix}^k. \tag{17}$$

The work in the composite structure due to bending arising from both normal and shear piezoelectrically induced stresses may be expressed as the area integral

$$W_b = \iint_A (M_1^P \kappa_1 + M_2^P \kappa_2 + M_6^P \kappa_6) \, dA, \tag{18}$$

where

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} -w_{xx} \\ -w_{yy} \\ -2w_{xy} \end{bmatrix} \tag{19}$$

are the curvatures. Substituting equations (19) and (16) into equation (18) yields an expression for the work due to the bending and torsion of the plate induced by the piezoelectric effect of the laminae:

$$W_b = -\iint_A \left[\sum_{k=1}^{n} e_{31}^{0k} z_0^k V^k w_{xx} + \sum_{k=1}^{n} e_{32}^{0k} z_0^k V^k w_{yy} + 2 \sum_{k=1}^{n} e_{36}^{0k} z_0^k V^k w_{xy} \right] \mathrm{d}A. \tag{20}$$

The work expression, W_s, due to the piezoelectric force resultants N_1^p, N_2^p and N_6^p may be written as

$$W_s = \iint_A [N_1^p u_x^0 + N_2^p v_y^0 + N_6^p (u_y^0 + v_x^0)] \, \mathrm{d}A. \tag{21}$$

Substituting equation (17) into equation (21) yields an expression for the work due to the piezoelectrically induced stretching of the plate:

$$W_x = \sum_{k=1}^{n} \iint_A [e_{31}^{0k} V^k u_x^0 + e_{32}^{0k} V^k v_y^0 + e_{36}^{0k} V^k (u_y^0 + v_x^0)] \, \mathrm{d}A. \tag{22}$$

The total work acting on the structure due to the combined piezoelectric effect of the laminae, W_p, is

$$W_p = W_b + W_s. \tag{23}$$

Substituting equations (20) and (22) into equation (23), carrying out the variation and integrating by parts yields an expression for the virtual work due solely to piezoelectrically induced forces:

$$\begin{aligned}\delta W_p = \sum_{k=1}^{n} \Bigg\{ \iint_A & [-(e_{31}^{0k} z_0^k V_{xx}^k + e_{32}^{0k} z_0^k V_{yy}^k + 2 e_{32}^{0k} z_0^k V_{xy}^k)\delta w + (e_{31}^{0k} V_x^k + e_{36}^{0k} V_y^k)\delta u \\
&+ (e_{32}^{0k} V_y^k + e_{36}^{0k} V_x^k)\delta v] \, \mathrm{d}A + \int_{-L_b/2}^{L_b/2} [e_{31}^{0k} z_0^k V_x^k \delta w - e_{31}^{0k} z_0^k V^k \delta w_x - e_{36}^{0k} z_0^k V^k \delta w_y \\
&+ e_{31}^{0k} V^k \delta u + e_{36}^{0k} V^k \delta v]_{x=-L_a/2}^{x=L_a/2} \, \mathrm{d}y + \int_{-L_a/2}^{L_a/2} [e_{32}^{0k} z_0^k V_x^k \delta w - e_{36}^{0k} z_0^k V^k \delta w_x \\
&- e_{32}^{0k} z_0^k V^k \delta w_y + e_{36}^{0k} V^k \delta u + e_{32}^{0k} V^k \delta v]_{y=-L_b/2}^{y=L_b/2} \, \mathrm{d}x \Bigg\}.\end{aligned} \tag{24}$$

2.2.2. *Virtual work due to dissipative elements*

In real systems which are physically constrained at the boundaries, dissipative boundary forces are present which extract energy from the system. Additionally, the absence of structural damping terms in the system model may be somewhat compensated for through the introduction of lumped dissipative elements at the boundaries. It is ascertained that the addition of structural damping in the model would lead to dissipative terms which are closely analogous to those which arise due to lumped damper elements. If it is assumed that the lumped damper elements produce forces which are proportional to velocity, then the virtual work due to the linear dissipative elements may be written as

$$\begin{aligned}\delta W_f = &-\int_{-L_b/2}^{L_b/2} [b_1 u_t^0 \delta u + b_2 v_t^0 \delta v + b_3 w_t \delta w + \beta_1 w_{xt} \delta w_x + \beta_2 w_{yt} \delta w_y]_{x=-L_a/2, L_a/2} \, \mathrm{d}y \\
&- \int_{-L_a/2}^{L_a/2} [b_1 u_t^0 \delta u + b_2 v_t^0 \delta v + b_3 w_t \delta w + \beta_1 w_{xt} \delta w_x + \beta_2 w_{yt} \delta w_y]_{y=-L_b/2, L_b/2} \, \mathrm{d}x,\end{aligned} \tag{25}$$

where the parameters b_1, b_2, b_3, β_1 and β_2 are positive and normalized per unit length.

2.2.3. *Kinetic and potential energy*

The total strain energy, $\mathscr{U}_b$ of a rectangular isotropic plate experiencing transverse bending and torsion is given by [17, 18, 21]

$$\mathscr{U}_b = \tfrac{1}{2}D \iint_A [w_{xx}^2 + 2\nu w_{xx} w_{yy} + w_{yy}^2 + 2(1-\nu)w_{xy}^2]\,\mathrm{d}A, \tag{26}$$

where D is the flexural rigidity of the plate and ν is the Poisson ratio. The total strain energy, $\mathscr{U}_s$ of an isotropic plate experiencing in-plane membrane stretching is given by [17, 18, 21]

$$\mathscr{U}_s = \tfrac{1}{2}A_0 \iint_A [(u_x^0)^2 + 2\nu u_x^0 v_y^0 + (v_y^0)^2 + \tfrac{1}{2}(1-\nu)(u_y^0 + v_x^0)^2]\,\mathrm{d}A, \tag{27}$$

where A_0 is the equivalent axial rigidity of the plate. The total strain energy $\mathscr{U}$, of an isotropic plate experiencing both bending and stretching is then a superposition of equations (26) and (27) (since there is no coupling effect in an isotropic continuum):

$$\mathscr{U} = \mathscr{U}_b + \mathscr{U}_s. \tag{28}$$

The total kinetic energy, T, of an isotropic plate experiencing both bending and stretching is

$$T = T_b + T_s, \tag{29}$$

where T_b and T_s are defined as

$$T_b = \tfrac{1}{2}\rho h \iint_A w_t^2\,\mathrm{d}A, \qquad T_s = \tfrac{1}{2}\rho h \iint_A [(u_t^0)^2 + (v_t^0)^2]\,\mathrm{d}A. \tag{30, 31}$$

2.2.4. *Equations of motion and boundary conditions*

Equations of motion and corresponding boundary conditions emerge through the substitution of the expressions for kinetic energy, potential energy and virtual work into the variational indicator (equation (13)). For convenience, force resultants and bending and twisting moment resultants due to the pure mechanical loading of an isotropic plate, as defined by Ashton and Whitney [17], are introduced

$$N_1^M = A_0(u_x^0 + \nu v_y^0), \qquad N_2^M = A_0(v_y^0 + \nu u_x^0), \qquad N_6^M = \tfrac{1}{2}A_0(1-\nu)(u_y^0 + v_x^0),$$
$$M_1^M = -D(w_{xx} + \nu w_{yy}), \qquad M_2^M = -D(w_{yy} + \nu w_{xx}), \qquad M_6^M = -D(1-\nu)w_{xy},$$
$$Q_1^M = -D(w_{xxx} + w_{xyy}), \qquad Q_2^M = -D(w_{yyy} + w_{xxy}). \tag{32}$$

Here the transverse shear force resultants are defined as [17]

$$(Q_1, Q_2) = \sum_{k=1}^{n} \int_{-h/2}^{h/2} ((\sigma^{xz})^k, (\sigma^{yz})^k)\,\mathrm{d}z. \tag{33}$$

Making use of the above definitions, substituting the energy and work expressions (equations (24)–(31)) into the variational indicator (equation (13)) and carrying out the variation yields

$$\sum_{k=1}^{n} \int_{t_1}^{t_2} \mathrm{d}t \bigg\{ \iint_A \mathrm{d}A(-[\rho h w_{tt} + D\nabla^4 w + e_{31}^{0k} z_0^k V_{xx}^k + 2e_{36}^{0k} z_0^k V_{xy}^k + e_{32}^{0k} z_0^k V_{yy}^k]\delta w$$

$$+ [A_0(u_{xx}^0 + \tfrac{1}{2}(1+\nu)v_{xy}^0 + \tfrac{1}{2}(1-\nu)u_{yy}^0) - \rho h u_{tt}^0 - e_{31}^{0k} V_x^k - e_{36}^{0k} V_y^k]\delta u$$

$$+ [A_0(v^0_{yy} + \tfrac{1}{2}(1+\nu)u^0_{xy} + \tfrac{1}{2}(1-\nu)v^0_{xx}) - \rho h v^0_{tt} - e^{0k}_{32}V^k_y - e^{0k}_{36}V^k_x]\delta v)$$

$$+ \int_{-L_b/2}^{L_b/2} \mathrm{d}y[(e^{0k}_{31}V^k - \alpha(b_1 u^0_t) - N^M_1)\delta u + (e^{0k}_{36}V^k - \alpha(b_2 v^0_t) - N^M_6)\delta v$$
$$- (e^{0k}_{31}z^k_0 V^k + \alpha(\beta_1 w_{xt}) - M^M_1)\delta w_x - (e^{0k}_{36}z^k_0 V^k + \alpha(\beta_2 w_{yt}) - M^M_6)\delta w_y$$
$$+ (e^{0k}_{31}z^k_0 V^k_x + e^{0k}_{36}z^k_0 V^k_y - \alpha(b_3 w_t) - Q^M_1)\delta w]^{x=L_a/2}_{x=-L_a/2}$$

$$+ \int_{-L_a/2}^{L_a/2} \mathrm{d}x[(e^{0k}_{32}V^k - \alpha(b_2 v^0_t) - N^M_2)\delta v + (e^{0k}_{36}V^k - \alpha(b_1 u^0_t) - N^M_6)\delta u$$
$$- (e^{0k}_{32}z^k_0 V^k + \alpha(\beta_2 w_{yt}) - M^M_2)\delta w_y - (e^{0k}_{36}z^k_0 V^k + \alpha(\beta_1 w_{xt}) - M^M_6)\delta w_x$$
$$+ (e^{0k}_{32}z^k_0 V^k_y + e^{0k}_{36}z^k_0 V^k_x - \alpha(b_3 w_t) - Q^M_2)\delta w]^{y=L_b/2}_{y=-L_b/2}\Bigg\}, \tag{34}$$

where α is defined as

$$\alpha = \begin{Bmatrix} +1 & \text{at } x = -L_a/2 \text{ or } y = -L_b/2 \\ -1 & \text{at } x = L_a/2 \text{ or } y = L_b/2 \end{Bmatrix}. \tag{35}$$

Equation (34) implies that for a rectangular isotropic laminated piezoelectric plate the following equations of motion must be satisfied:

$$A_0(u^0_{xx} + \tfrac{1}{2}(1+\nu)v^0_{xy} + \tfrac{1}{2}(1-\nu)u^0_{yy}) = \rho h u^0_{tt} + \sum_{k=1}^{n} e^{0k}_{31}V^k_x + \sum_{k=1}^{n} e^{0k}_{36}V^k_y, \tag{36}$$

$$A_0(v^0_{yy} + \tfrac{1}{2}(1+\nu)u^0_{xy} + \tfrac{1}{2}(1-\nu)v^0_{xx}) = \rho h v^0_{tt} + \sum_{k=1}^{n} e^{0k}_{32}V^k_y + \sum_{k=1}^{n} e^{0k}_{36}V^k_x, \tag{37}$$

$$D\nabla^4 w + \rho h w_{tt} = -\sum_{k=1}^{n} e^{0k}_{31}z^k_0 V^k_{xx} - 2\sum_{k=1}^{n} e^{0k}_{36}z^k_0 V^k_{xy} - \sum_{k=1}^{n} e^{0k}_{32}z^k_0 V^k_{yy}, \tag{38}$$

subject to the boundary conditions given in Table 1. The boundary conditions as described in the table are *identical* to those given in reference [13] for the special case of b_1, b_2, b_3, β_1, $\beta_2 = 0$ (no dissipative losses at the boundaries). Note that of course the mechanical force and moment resultants (equations (32)) must be generalized. The exact interpretation

TABLE 1

Boundary conditions for an anisotropic rectangular plate

At $x = -L_a/2$ and $x = L_a/2$		At $y = -L_b/2$ and $y = L_b/2$	
$N^M_1 - \sum_{k=1}^{n} e^{0k}_{31}V^k + \alpha(b_1 u^0_t)$	or u^0	$N^M_2 - \sum_{k=1}^{n} e^{0k}_{32}V^k + \alpha(b_2 v^0_t)$	or v^0
$N^M_6 - \sum_{k=1}^{n} e^{0k}_{36}V^k + \alpha(b_2 v^0_t)$	or v^0	$N^M_6 - \sum_{k=1}^{n} e^{0k}_{36}V^k + \alpha(b_1 u^0_t)$	or u^0
$Q^M_1 - \sum_{k=1}^{n} e^{0k}_{31}z^k_0 V^k_x$		$Q^M_2 - \sum_{k=1}^{n} e^{0k}_{32}z^k_0 V^k_y$	
$- \sum_{k=1}^{n} e^{0k}_{36}z^k_0 V^k_y + \alpha(b_3 w_t)$	or w	$- \sum_{k=1}^{n} e^{0k}_{36}z^k_0 V^k_x + \alpha(b_3 w_t)$	or w
$M^M_1 - \sum_{k=1}^{n} e^{0k}_{31}z^k_0 V^k - \alpha(\beta_1 w_{xt})$	or w_x	$M^M_2 - \sum_{k=1}^{n} e^{0k}_{32}z^k_0 V^k - \alpha(\beta_2 w_{yt})$	or w_y
$M^M_6 - \sum_{k=1}^{n} e^{0k}_{36}z^k_0 V^k - \alpha(\beta_2 w_{yt})$	or w_y	$M^M_6 - \sum_{k=1}^{n} e^{0k}_{36}z^k_0 V^k - \alpha(\beta_1 w_{xt})$	or w_x

of these resultants for the anisotropic system will become evident in the section to follow. It is worthwhile to note that the equations of motion for the isotropic plate (equations (36)–(38)) are a special case of the anisotropic plate results equations (1)–(3)). The boundary conditions and the general (anisotropic) equations of motion (equations (1)–(3)) will be incorporated into an energy flux expression derived in the following section, leading to the establishment of stability criteria and a control strategy for a general anisotropic system.

2.3. LYAPUNOV ENERGY FUNCTIONAL

Stability criteria are established based on a Lyapunov functional, J, which is chosen to be representative of the total system energy. The criteria that emerge ensure that the functional total time derivative will be negative definite, and consequently guarantee that the system will be asymptotically stable. The Lyapunov functional to be considered is defined as

$$J = \tfrac{1}{2} \iint_A [\rho h(w_t^2 + (u_t^0)^2 + (v_t^0)^2) + \mathscr{U}_b + \mathscr{U}_s + \mathscr{U}_c]\, \mathrm{d}A, \tag{39}$$

where $\mathscr{U}_b$ is the total strain energy due to pure bending and torsion, $\mathscr{U}_s$ is the total strain energy due to pure stretching, and $\mathscr{U}_c$ consists of the strain energy terms which arise due to the anisotropic coupling between stretching and bending. The strain energy terms may be expressed as [17]

$$\mathscr{U}_b = D_{11} w_{xx}^2 + 2D_{12} w_{xx} w_{yy} + D_{22} w_{yy}^2 + 4D_{16} w_{xx} w_{xy} + 4D_{26} w_{yy} w_{xy} + 4D_{66} w_{xy}^2, \tag{40}$$

$$\begin{aligned} \mathscr{U}_s = {} & A_{11}(u_x^0)^2 + 2A_{12} u_x^0 v_y^0 + A_{22}(v_y^0)^2 + 2A_{16}(u_x^0 u_y^0 + u_x^0 v_x^0) \\ & + 2A_{26}(u_y^0 v_y^0 + v_x^0 v_y^0) + A_{66}(u_y^0 + v_x^0)^2, \end{aligned} \tag{41}$$

$$\begin{aligned} \mathscr{U}_c = {} & -2B_{11} u_x^0 w_{xx} - 2B_{12}(v_y^0 w_{xx} + u_x^0 w_{yy}) - 2B_{22} v_y^0 w_{yy} - 2B_{16}(u_y^0 w_{xx} + v_x^0 w_{xx} + 2u_x^0 w_{xy}) \\ & - 4B_{66}(u_y^0 w_{xy} + v_x^0 w_{xy}). \end{aligned} \tag{42}$$

Asymptotic stability is guaranteed when the time derivative of equation (39) is negative definite: i.e., when

$$\dot{J} = \iint_A [\rho h(w_{tt} w_t + u_{tt}^0 u_t^0 + v_{tt}^0 v_t^0) + \tfrac{1}{2}(\dot{\mathscr{U}}_b + \dot{\mathscr{U}}_s + \dot{\mathscr{U}}_c)]\, \mathrm{d}A < 0. \tag{43}$$

2.3.1. *Bending strain energy rate*

In order to simplify equation (43), each of the strain energy terms will be considered separately. The $\dot{\mathscr{U}}_b$ term due to the pure bending strain energy may be expanded, in consideration of equation (40), as

$$\begin{aligned} \tfrac{1}{2} \iint_A \dot{\mathscr{U}}_b\, \mathrm{d}A = \iint_A [& D_{11} w_{xx} w_{xxt} + D_{12}(w_{xx} w_{yyt} + w_{yy} w_{xxt}) + D_{22} w_{yy} w_{yyt} \\ & + 2D_{16}(w_{xx} w_{xyt} + w_{xy} w_{xxt}) + 2D_{26}(w_{yy} w_{xyt} + w_{xy} w_{yyt}) \\ & + 4D_{66} w_{xy} w_{xyt}]\, \mathrm{d}A. \end{aligned} \tag{44}$$

For a regular orthotropic plate in bending, the bending moment and shear force resultants are defined as [17]

$$M_1^B = -(D_{11} w_{xx} + D_{12} w_{yy} + 2D_{16} w_{xy}), \qquad M_2^B = -(D_{22} w_{yy} + D_{12} w_{xx} + 2D_{26} w_{xy}),$$
$$M_6^B = -(2D_{66} w_{xy} + D_{16} w_{xx} + D_{26} w_{yy}),$$

$$Q_1^B = -[D_{11}w_{xxx} + (D_{12} + 2D_{66})w_{xyy} + 3D_{16}w_{xxy} + D_{26}w_{yyy}],$$

$$Q_2^B = -[D_{22}w_{yyy} + (D_{12} + 2D_{66})w_{xxy} + 3D_{26}w_{xyy} + D_{16}w_{xxx}]. \tag{45}$$

Integrating equation (44) by parts and applying the definitions found in equations (10) and (45) yields

$$\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_b \,\mathrm{d}A = \iint_A \mathscr{L}_b w_t \,\mathrm{d}A + \int_{-L_b/2}^{L_b/2})[Q_1^B w_t - M_1^B w_{xt} - M_6^B w_{yt}]_{x=-L_a/2}^{x=L_a/2} \,\mathrm{d}y$$

$$+ \int_{-L_a/2}^{L_a/2} [Q_2^B w_t - M_2^B w_{yt} - M_6^B w_{xt}]_{y=-L_b/2}^{y=L_b/2} \,\mathrm{d}x. \tag{46}$$

2.3.2. *Longitudinal strain energy rate*

The $\dot{\mathscr{U}}_s$ term due to pure longitudinal and in-plane shear strain in equation (43) may be similarly reduced. Taking the time derivative of equation (41) and integrating over the area yields

$$\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_s \,\mathrm{d}A = \iint_A [A_{11}u_x^0 u_{xt}^0 + A_{12}(u_x^0 v_{yt}^0 + v_y^0 u_{xt}^0) + A_{22}v_y^0 v_{yt}^0$$

$$+ A_{16}(u_x^0 u_{yt}^0 + u_y^0 u_{xt}^0 + u_x^0 v_{xt}^0 + v_x^0 u_{xt}^0)$$

$$+ A_{26}(v_y^0 u_{yt}^0 + u_y^0 v_{yt}^0 + v_y^0 v_{xt}^0 + v_x^0 v_{yt}^0)$$

$$+ A_{66}(u_y^0 + v_x^0)(u_{yt}^0 + v_{xt}^0)] \,\mathrm{d}A. \tag{47}$$

For a regular orthotropic plate experiencing pure longitudinal and in-plane shear strain, the force resultants acting on the body are defined as [17]

$$N_1^S = A_{11}u_x^0 + A_{12}v_y^0 + A_{16}(u_y^0 + v_x^0), \qquad N_2^S = A_{12}u_x^0 + A_{22}v_y^0 + A_{26}(u_y^0 + v_x^0),$$

$$N_6^S = A_{16}u_x^0 + A_{26}v_y^0 + A_{66}(u_y^0 + v_x^0). \tag{48}$$

Integrating equation (47) by parts and substituting equations (4), (5) and (48) into the result allows the strain rate integral $\tfrac{1}{2}\int\int_A \dot{\mathscr{U}}_s \,\mathrm{d}A$ to be expressed as

$$\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_s \,\mathrm{d}A = -\iint_A [\mathscr{L}_{s1}u_t^0 + \mathscr{L}_{s2}v_t^0] \,\mathrm{d}A + \int_{-L_b/2}^{L_b/2} [N_1^S u_t^0 + N_6^S v_t^0]_{x=-L_a/2}^{x=L_a/2} \,\mathrm{d}y$$

$$+ \int_{-L_a/2}^{L_a/2} [N_2^S v_t^0 + N_6^S u_t^0]_{y=-L_b/2}^{y=L_b/2} \,\mathrm{d}x. \tag{49}$$

2.3.3. *Coupled strain energy rate*

The coupling strain rate integral $\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_c \,\mathrm{d}A$ found in equation (43) is significantly more difficult to simplify, since all of its terms are products of transverse and axial displacement derivatives. In the light of equation (42), the coupled strain energy rate may be written as

$$\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_c \,\mathrm{d}A = \iint_A (\dot{\mathscr{U}}_{11} + \dot{\mathscr{U}}_{12} + \dot{\mathscr{U}}_{22} + \dot{\mathscr{U}}_{16} + \dot{\mathscr{U}}_{26} + \dot{\mathscr{U}}_{66}) \,\mathrm{d}A, \tag{50}$$

where $\mathscr{U}_{11}$, $\mathscr{U}_{12}$, $\mathscr{U}_{22}$, $\mathscr{U}_{16}$, $\mathscr{U}_{26}$ and $\mathscr{U}_{66}$ are defined as follows:

$$\mathscr{U}_{11} = -B_{11}u_x^0 w_{xx}, \qquad \mathscr{U}_{12} = -B_{12}(v_y^0 w_{xx} + u_x^0 w_{yy}), \qquad \mathscr{U}_{22} = -B_{22}v_y^0 w_{yy},$$

$$\mathscr{U}_{16} = -B_{16}(u_y^0 w_{xx} + v_x^0 w_{xx} + 2u_x^0 w_{xy}), \qquad \mathscr{U}_{26} = -B_{26}(u_y^0 w_{yy} + v_x^0 w_{yy} + 2v_y^0 w_{xy}),$$

$$\mathscr{U}_{66} = -B_{66}(u_y^0 w_{xy} + v_x^0 w_{xy}). \tag{51}$$

Integrating equation (50) by parts and making use of equations (6)–(9) allows the coupled strain energy rate to be expressed as [22]

$$\tfrac{1}{2}\iint_A \dot{\mathscr{U}}_c \, \mathrm{d}A = \iint_A [\mathscr{L}_{c1}u_t^0 + \mathscr{L}_{c2}v_t^0 - (\mathscr{L}_{c3} + \mathscr{L}_{c4})w_t]\, \mathrm{d}A$$
$$+ \int_{-L_b/2}^{L_b/2} [Q_1^C w_t - M_1^C w_{xt} - M_6^C w_{yt} + N_1^C u_t^0 + N_6^C v_t^0]_{x=-L_a/2}^{x=L_a/2}\, \mathrm{d}y$$
$$+ \int_{-L_a/2}^{L_a/2} [Q_2^C w_t - M_2^C w_{yt} - M_6^C w_{xt} + N_2^C v_t^0 + N_6^C u_t^0]_{y=-L_b/2}^{y=L_b/2}\, \mathrm{d}x, \qquad (52)$$

where the following definitions have been used:

$$N_1^C = -B_{11}w_{xx} + 2B_{16}w_{xy} + B_{12}w_{yy}, \qquad N_2^C = -B_{12}w_{xx} + 2B_{26}w_{xy} + B_{22}w_{yy},$$
$$N_6^C = -B_{16}w_{xx} + 2B_{66}w_{xy} + B_{26}w_{yy},$$
$$M_1^C = B_{11}u_x^0 + B_{12}v_y^0 + B_{16}(u_y^0 + v_x^0), \qquad M_2^C = B_{12}u_x^0 + B_{22}v_y^0 + B_{26}(u_y^0 + v_x^0),$$
$$M_6^C = B_{16}u_x^0 + B_{26}v_y^0 + B_{66}(u_y^0 + v_x^0),$$
$$Q_1^C = B_{11}u_{xx}^0 + (B_{12} + B_{66})v_{xy}^0 + B_{66}u_{yy}^0 + 2B_{16}u_{xy}^0 + B_{16}v_{xx}^0 + B_{26}v_{yy}^0,$$
$$Q_2^C = B_{22}v_{yy}^0 + (B_{12} + B_{66})u_{xy}^0 + B_{66}v_{xx}^0 + B_{16}u_{xx}^0 + 2B_{26}v_{xy}^0 + B_{26}u_{yy}^0. \qquad (53)$$

2.3.4. *General system energy flux*

The general system energy flux expression (43) may be greatly simplified through combining the major expressions given in the preceding sections. For a general anisotropic plate, the moment and force resultants which arise due to the mechanical loading of a general anisotropic plate can be written as [17]

$$N_1^M = N_1^S + N_1^C, \qquad N_2^M = N_2^S + N_2^C, \qquad N_6^M = N_6^S + N_6^C,$$
$$M_1^M = M_1^B + M_1^C, \qquad M_2^M = M_2^B + M_2^C, \qquad M_6^M = M_6^B + M_6^C,$$
$$Q_1^M = Q_1^B + Q_1^C, \qquad Q_2^M = Q_2^B + Q_2^C. \qquad (54)$$

Making use of these definitions and substituting equations (46), (49) and (52) into equation (43) yields

$$\dot{J} = \iint_A [(\rho h w_{tt} + \mathscr{L}_b - \mathscr{L}_{c3} - \mathscr{L}_{c4})w_t + (\rho h u_{tt}^0 - \mathscr{L}_{s1} + \mathscr{L}_{c1})u_t^0$$
$$+ (\rho h v_{tt}^0 - \mathscr{L}_{s2} + \mathscr{L}_{c2})v_t^0]\, \mathrm{d}A + \int_{-L_b/2}^{L_b/2} [Q_1^M w_t - M_1^M w_{xt}$$
$$- M_6^M w_{yt} + N_1^M u_t^0 + N_6^M v_t^0]_{x=-L_a/2}^{x=L_a/2}\, \mathrm{d}y + \int_{-L_a/2}^{L_a/2} [Q_2^M w_t$$
$$- M_2^M w_{yt} - M_6^M w_{xt} + N_2^M v_t^0 + N_6^M u_t^0]_{y=-L_b/2}^{y=L_b/2}\, \mathrm{d}x. \qquad (55)$$

Incorporating the equations of motion (1)–(3) into the preceding expression gives

$$\dot{J} = -\sum_{k=1}^{n} \left\{ \iint_A [(e_{31}^{0k} z_0^k V_{xx}^k + 2e_{36}^{0k} z_0^k V_{xy}^k + e_{32}^{0k} z_0^k V_{yy}^k)w_t + (e_{31}^{0k} V_x^k + e_{36}^{0k} V_y^k)u_t^0\right.$$
$$\left. + (e_{36}^{0k} V_x^k + e_{32}^{0k} V_y^k)v_t^0]\, \mathrm{d}A \right\} + \int_{-L_b/2}^{L_b/2} [Q_1^M w_t - M_1^M w_{xt} - M_6^M w_{yt} + N_1^M u_t^0$$

$$+ N_6^M v_t^0]_{x=-L_a/2}^{x=L_a/2} \,\mathrm{d}y + \int_{-L_a/2}^{L_a/2} [Q_2^M w_t - M_2^M w_{yt} - M_6^M w_{xt} + N_2^M v_t^0$$

$$+ N_6^M u_t^0]_{y=-L_b/2}^{y=L_b/2} \,\mathrm{d}x, \tag{56}$$

where the integral expression now includes only the piezoelectric effect of the n actuator layers. Integrating equation (56) by parts leads to the expression

$$\dot{J} = -\sum_{k=1}^{n} \iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0 - e_{36}^{0k}$$

$$\times (u_{yt}^0 + v_{xt}^0)] V^k \,\mathrm{d}A + \sum_{k=1}^{n} \int_{-L_b/2}^{L_b/2} [(Q_1^M - e_{31}^{0k} z_0^k V_x^k - e_{36}^{0k} z_0^k V_y^k) w_t$$

$$- (M_1^M - e_{31}^{0k} z_0^k V^k) w_{xt} - (M_6^M - e_{36}^{0k} z_0^k V^k) w_{yt} + (N_1^M - e_{31}^{0k} V^k) u_t^0$$

$$+ (N_6^M - e_{36}^{0k} V^k) v_t^0]_{x=-L_a/2}^{x=L_a/2} \,\mathrm{d}y + \sum_{k=1}^{n} \int_{-L_a/2}^{L_a/2} [(Q_2^M - e_{32}^{0k} z_0^k V_x^k - e_{36}^{0k} z_0^k V_x^k) w_t$$

$$- (M_2^M - e_{32}^{0k} z_0^k V^k) w_{yt} - (M_6^M - e_{36}^{0k} z_0^k V^k) w_{xt} + (N_2^M - e_{32}^{0k} V^k) v_t^0$$

$$+ (N_6^M - e_{36}^{0k} V^k) u_t^0]_{y=-L_b/2}^{y=L_b/2} \,\mathrm{d}x. \tag{57}$$

If the plate displacements and velocities are unconstrained at the boundaries and boundary lumped damper terms are retained, than applying the *boundary conditions* (see Table 1) to equation (57) yields

$$\dot{J} = -\sum_{k=1}^{n} \iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0 - e_{36}^{0k}$$

$$\times (u_{yt}^0 + v_{xt}^0)] V^k \,\mathrm{d}A - \int_{-L_b/2}^{L_b 2} [b_1 (u_t^0)^2 + b_2 (v_t^0)^2 + b_3 w_t^2$$

$$+ \beta_1 w_{xt}^2 + \beta_2 w_{yt}^2]_{x=-L_a/2, L_a/2} \,\mathrm{d}y - \int_{-L_a/2}^{L_a/2} [b_1 (u_t^0)^2 + b_2 (v_t^0)^2 + b_3 w_t^2$$

$$+ \beta_1 w_{xt}^2 + \beta_2 w_{yt}^2]_{y=-L_b/2, L_b/2} \,\mathrm{d}x. \tag{58}$$

If the Lyapunov functional time derivative is negative definite, then the system is asymptotically stable. For virtually any combination of clamped, pinned, free or sliding boundary conditions, one or more of the dissipative terms in equation (58) will be non-vanishing. Hence the latter two boundary integrals in equation (58) are both positive definite when the system is non-static. The negative definiteness of $\dot{J}$ is thus ensured if the first integral expression is negative semidefinite. Thus the sufficient condition which emerges to guarantee asymptotic stability is

$$\dot{J}_p \leqslant 0, \tag{59}$$

where

$$\dot{J}_p = -\sum_{k=1}^{n} \iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0$$

$$- e_{36}^{0k} (u_{yt}^0 + v_{xt}^0)] V^k \,\mathrm{d}A. \tag{60}$$

The validity of expression (59) as a sufficient condition for asymptotic stability is supported through physical insight. Since the Lyapunov functional was originally based on the total mechanical energy, equation (58) represents the system energy flux. The piezoelectric laminate control voltages, $V^k(x, y, t)$, appear in equation (58) only through the single integral expression given in equation (60). If the control voltages are set to zero, then the system is passive. However, any real laminated plate system is characterized by dissipative losses (structural damping, interlaminate friction, non-ideal boundary constraints, etc.) which ensure that the system energy flux of the non-static passive system is still negative definite. As long as the active piezoelectric elements do not *add* energy to the system, then asymptotic stability is ensured. The measure of active vibration suppression achieved in any given design depends solely on choosing appropriate control voltage functions which cause $\dot{J}_p$ to be large in magnitude and negative in sign so as to extract energy rapidly. Physical arguments dictate that a higher fidelity model which includes non-conservative loss mechanisms such as structural damping should lead to an energy flux expression similar to equation (58), in which the dissipative terms are always negative and non-vanishing regardless of boundary conditions.

2.4. STABILITY CRITERIA

Sufficient conditions on the actuator laminae control inputs that ensure asymptotic stability follow from expressions (59) and (60). The control input voltage of the kth layer, $V^k(x, y, t)$, is assumed to be separable into spatial and temporal functions: i.e.,

$$V^k(x, y, t) = V_0^k \Lambda^k(x, y)\rho^k(t), \tag{61}$$

where V_0^k is the maximum voltage amplitude applied to the film and $\Lambda^k(x, y)$ and $\rho^k(t)$ are the non-dimensional spatial distribution and the time-variant control input functions, respectively, of the kth piezoelectric actuation layer. $\Lambda^k(x, y)$ may be accomplished in practice through varying the distribution of the surface electrode or by spatially varying the layer thickness [3, 4, 11]. Combining equation (61) with equation (60) gives

$$\dot{J}_p = -\sum_{k=1}^{n} V_0^k \rho^k(t) \iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0 - e_{36}^{0k}(u_{yt}^0 + v_{xt}^0)]\Lambda^k(x, y)\,\mathrm{d}A. \tag{62}$$

A general and sufficient condition for stability implied by this statement is that the set of input control functions, $\rho^k(t)$, must ensure the negative semidefiniteness of $\dot{J}_p$. However, it will generally be more practical to express a stronger condition on each particular actuator layer such that if this condition is satisfied for all n actuator layers then asymptotic stability is ensured. Specifically, $\dot{J}_p$ *must* be negative semidefinite if, for all k from 1 to n,

$$\rho^k(t) = \mathrm{sgn} \iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0 - e_{36}^{0k}(u_{yt}^0 + v_{xt}^0)]\Lambda^k(x, y)\,\mathrm{d}A. \tag{63}$$

Equation (63) is typically the most general condition which will be of practical consequence in the control design process. If equation (63) is satisfied and all $\Lambda^k(x, y)$ are chosen so that the expression is non-zero, then some measure of active vibration attenuation is necessarily ensured, since energy is actively extracted from the system (i.e., $\dot{J}_p < 0$). If the choices for $\Lambda^k(x, y)$ cause the integral sum to vanish, then active vibration suppression is lost, but asymptotic stability is still ensured provided that dissipative forces act at the boundaries (since $\dot{J}_p$ will be negative definite). Controller effectiveness may be weighted in favor of individual modes, modal subsets or all modes of a system through a proper choice of spatial weighting functions, $\Lambda^k(x, y)$. The viability of spatially weighting the control authority has been demonstrated successfully for beams [3] and plates [14], and represents a broad topic suitable for future analysis.

3. VIBRATION CONTROL DESIGN STRATEGY

3.1. ANISOTROPIC PLATE CONTROL DESIGN

One may exploit the duality which exists between piezoelectric spatially distributed sensors and actuators in order to arrive at certain easily implementable constraints which will be sufficient to ensure both stability and the active dissipation of vibrational energy. The current accumulated on the surface electrode of the kth lamina due to the mechanical displacement of the laminates is [13]

$$i^k(t) = -\iint_A [e_{31}^{0k} z_0^k w_{xxt} + e_{32}^{0k} z_0^k w_{yyt} + 2e_{36}^{0k} z_0^k w_{xyt} - e_{31}^{0k} u_{xt}^0 - e_{32}^{0k} v_{yt}^0 - e_{36}^{0k}(u_{yt}^0 + v_{xt}^0)]\Lambda^k(x, y)\, \mathrm{d}A, \quad (64)$$

where $i^k(t)$ is the current measured through the kth electrode. It will become convenient to rewrite equations (62) and (64) respectively in the forms

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k \rho^k(t)[\bar{P}^k - z_0^k \bar{B}^k], \qquad i^k(t) = \bar{P}^k - z_0^k \bar{B}^k, \quad (65, 66)$$

where

$$\bar{P}^k = \iint_a [e_{31}^{0k} u_{xt}^0 + e_{32}^{0k} v_{yt}^0 + e_{36}^{0k}(u_{yt}^0 + v_{xt}^0)]\Lambda^k(x, y)\, \mathrm{d}A, \quad (67)$$

$$\bar{B}^k = \iint_A [e_{31}^{0k} w_{xxt} + e_{32}^{0k} w_{yyt} + 2e_{36}^{0k} w_{xyt}]\Lambda^k(x, y)\, \mathrm{d}A. \quad (68)$$

$\bar{P}^k$ contains all those terms which are relevant to the pure stretching and shearing of the structure, while $\bar{B}^k$ is related to structural bending and torsion.

Combining equations (65) and (66) leads to

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k \rho^k(t) i^k(t), \quad (69)$$

from which it becomes obvious that stability would be guaranteed if

$$\rho^k(t) = -\mathrm{sgn}\,[i^k(t)]. \quad (70)$$

Unfortunately, the enforcement of the above statement is not easily realizable, since it implies that each laminate must function identically both as a sensor and as an actuator. An alternative design strategy will therefore be developed which will allow each layer to assume a single function.

Previous studies involving bending vibration control of isotropic beams [10, 11] and plates [12] have shown that in such circumstances it is sufficient to ensure stability and active vibration attenuation provided that (1) each actuator layer is associated with an identically spatially varying sensor layer which is symmetrically co-located on the opposite side of the mid-plane, and (2) the actuator control inputs are always identical in sign to the current induced by the corresponding sensors. This design strategy fails, however, when the structure possesses asymmetry. Enforcing such a law implies that if the midplane of the kth actuator layer is located at a distance z_a^k above the mid-plane, then the kth sensor is located at $-z_a^k$. For pure bending motions the linear control law is

$$\rho^k(t) = z_a^k \bar{B}^k, \tag{71}$$

since the current generated in the corresponding sensor is

$$i^k(t) = z_a^k \bar{B}^k. \tag{72}$$

Thus, for pure bending $\dot{J}_p$, (equation (69)) is negative definite and the control strategy succeeds. However, if the plate undergoes both bending and stretching then this design strategy leads to the control law

$$\rho^k(t) = \bar{P}^k + z_a^k \bar{B}^k. \tag{73}$$

Inserting equation (73) into equation (65) gives

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k (\bar{P}^k + z_a^k \bar{B}^k)(\bar{P}^k - z_a^k \bar{B}^k), \tag{74}$$

which reduces to

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k [\bar{P}^{k^2} - (z_a^k)^2 \bar{B}^{k^2}]. \tag{75}$$

$\dot{J}_p$ is thus positive if the magnitude of $\bar{P}^k$ is greater than the magnitude of $z_a^k \bar{B}^k$. Significant membrane stretching therefore potentially causes this design approach to be destabilizing.

A simple design strategy for an asymmetric structure in which stability is guaranteed may be developed through the consideration of a structure containing n actuator/sensor *pairs* above the mid-plane and n pairs symmetrically co-located below the mid-plane. The actuator and sensor spatial polarization profiles in any given pair are identical both to each other and to the profiles corresponding to the co-located pair on the opposite side of the mid-plane. The kth actuator and sensor layers are located, respectively, at arbitrary heights z_a^k and z_s^k. The contribution of the $2n$ piezoelectric actuators to the energy flux is

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k \rho^k(t) [\bar{P}^k - z_a^k \bar{B}^k] + \sum_{k=1}^{n} V_1^k \phi^k(t) [\bar{P}^k + z_a^k \bar{B}^k], \tag{76}$$

where the first summation corresponds to actuator layers above the mid-plane and the latter to actuator layers below the mid-plane. The control gains of symmetrically co-located actuators are represented as V_0^k and V_1^k respectively. If the control input to each actuator layer is proportional and opposite in sign to the current of its paired sensor layer, then the control laws are

$$\rho^k(t) = -(\bar{P}^k - z_s^k \bar{B}^k), \qquad \phi^k(t) = -(\bar{P}^k + z_s^k \bar{B}^k). \tag{77, 78}$$

Substituting equation (78) into equation (76) gives

$$\dot{J}_p = -\sum_{k=1}^{n} [V_0^k (\bar{P}^k - z_s^k \bar{B}^k)(\bar{P}^k - z_a^k \bar{B}^k) + V_1^k (\bar{P}^k + z_s^k \bar{B}^k)(\bar{P}^k + z_a^k \bar{B}^k)]. \tag{79}$$

Expanding the expression yields

$$\dot{J}_p = -\sum_{k=1}^{n} [(V_0^k + V_1^k)\bar{P}^{k^2} + (V_1^k - V_0^k)(z_s^k + z_a^k)\bar{B}^k \bar{P}^k + (V_0^k + V_1^k) z_s^k z_a^k \bar{B}^{k^2}]. \tag{80}$$

If the control gains are established such that $V_0^k = V_1^k$, then equation (80) reduces to

$$\dot{J}_p = -2 \sum_{k=1}^{n} V_0^k [\bar{P}^{k^2} + z_s^k z_a^k \bar{B}^{k_2}]. \tag{81}$$

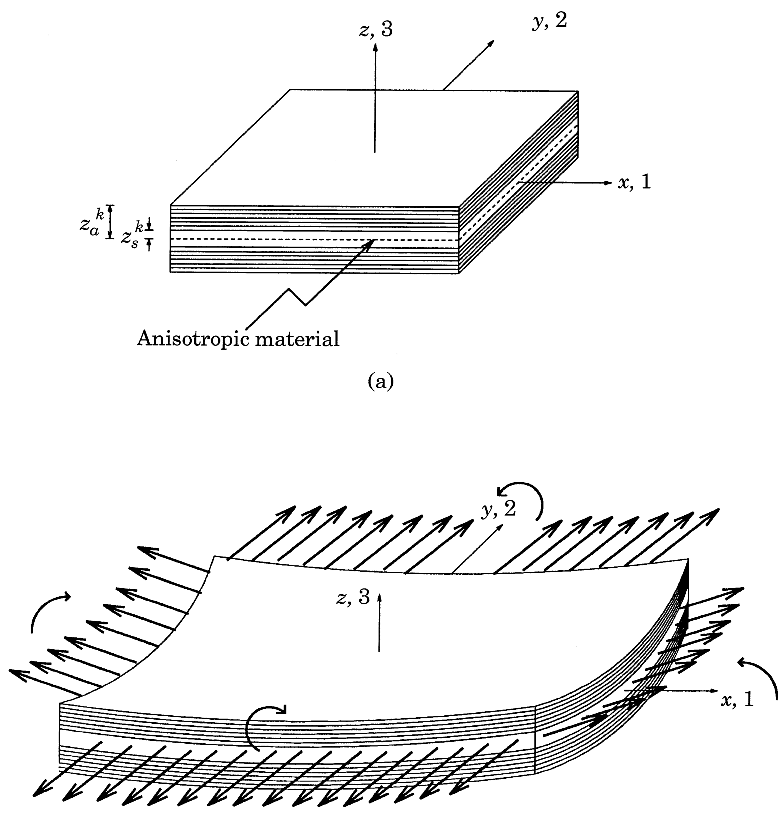
Since the control gains are always positive and z_a^k and z_s^k are identical in sign by definition, $\dot{J}_p$ must be negative semidefinite. The inclusion of boundary lumped damper terms causes the total time derivative of the Lyapunov function (equation (58)) to be negative definite: asymptotic stability is thus guaranteed. Upon recalling that $\dot{J}_p$ is proportional to the contribution of the piezoelectric layers to the total energy flux, it is then understood that if the spatial weighting functions, $\Lambda^k(x, y)$, are chosen such that equation (81) is always negative then some measure of active vibration suppression is ensured.

The physical basis of this control strategy is understood through considering the action of a single sensor/actuator pair located above the plate mid-plane at altitudes z_s^k and z_a^k, respectively. The pair's resulting (piezoelectrically induced) energy flux contribution, $\dot{J}_p^k$, becomes

$$\dot{J}_p^k = -V_0^k[\bar{P}^{k2} - (z_s^k + z_a^k)\bar{B}^k\bar{P}^k + z_s^k z_a^k\bar{B}^{k2}]. \tag{82}$$

The anisotropic behavior of the plate induces a destablizing energy term which appears as the second term in the above expansion. The term arises from the coupling between bending–torsion and stretching–shearing. However, the coupling term is completely eliminated by an identical sensor and actuator pair collocated on the opposite side of the neutral mid-plane, as seen in equation (80).

Figure 3 helps to lend further physical insight into the control law. Consider the sensor/actuator pair located above the mid-plane at altitudes z_s^k and z_a^k, respectively.



(a)

(b)

Figure 3. An anisotropic laminated plate undergoing bending and stretching. (a) Equilibrium state; (b) bending–stretching excitation state.

Suppose that at any given time the strain rates u^0_{xt}, u^0_{yt}, v^0_{xt}, w_{xxt}, w_{yyt} and w_{xyt} are all positive. For simplicity, it will be assumed that the laminate skew angles (see Figure 1) are zero, so that the e_{36} terms in equation (67) and (68) are vanishing. If z^k_s is sufficiently small then the current generated by the sensor layer will be positive. Applying the negative feedback law causes the control signal into the actuator to be negative. Thus the actuator layer will contract, causing stretching to be reduced while bending is increased. In contrast, consider the identical sensor/actuator pair on the opposite side of the mid-plane. The sensor layer located at $-z^k_s$ also generates a positive current. The resulting negative control input into its corresponding actuator will thus reduce not only stretching but also bending in the plate. However, since the control law is linear and proportional to $\bar{P}^k - z^k_s\bar{B}^k$, the magnitude of the signal into the actuator located at $+z^k_a$ will be smaller than the actuator located at $-z^k_a$. The destabilizing effect of the upper pair is thus negated by the lower pair, leading to a net decrease in the energy flux which is equivalent to the quantity expressed in equation (81).

A potential limitation of the proposed design approach is that $\dot{J}_p$ is guaranteed to be negative semidefinite only if the control gains of both sensor/actuator pairs are identical. Any mismatch in control gains causes the coupling term in equation (80) to be non-vanishing, and $\dot{J}_p$ is negative semidefinite only if

$$\sum_{k=1}^{n} (V^k_1 - V^k_0)(z^k_s + z^k_a)\bar{P}^k\bar{B}^k \leqslant \sum_{k=1}^{n} (V^k_0 + V^k_1)[\bar{P}^{k2} + z^k_s z^k_a \bar{B}^{k2}]. \tag{83}$$

If, for all n transducer pairs, $(V^k_1 - V^k_0)$ and $\bar{P}^k\bar{B}^k$ are different in sign, then the above condition is automatically met and the system is asymptotically stable. However, if for any given pair they are of the same sign then condition (83) may or may not be satisfied. The smaller the difference between the gains, the more easily satisfied is condition (83) and thus the more robust is the system mismatching of the control gains. In general, small mismatching errors should not affect the overall design. The robustness of the control method to the control gain mismatching of conjugate pairs should be more fully addressed in a future study.

While other design strategies are possible, this particular methodology is easily implemented in practice. The constraints enforced in this design are *sufficient* conditions to guarantee global stability. If spatial transducer polarization functions are chosen so that $\dot{J}_p < 0$, then the active extraction of vibrational energy is ensured. In summary, three criteria were established in the design process. First, for each piezoelectric actuator laminate above the composite structure mid-plane there must exist a corresponding identically polarized sensor laminate located above the mid-plane. Second, the control law governing each conjugate sensor/actuator pair is constrained such that the input to a given actuator is always proportional and opposite in sign to the current induced by the corresponding sensor. Last, for each conjugate pair above the mid-plane there exists an identical pair below the mid-plane. These criteria ensure the negative semidefiniteness of the energy flux expression, while a suitable choice in $\Lambda^k(x, y)$ will ensure that the system is actively dissipative.

3.2. RELAXATION OF CONSTRAINTS FOR SPECIAL CASES

If the system is orthotropic (or isotropic) such that bending–stretching coupling is absent, then these design constraints may be relaxed significantly. If such a system experiences pure bending and twisting only, then $\bar{P}^k$ vanishes and the piezoelectric energy flux, $\dot{J}_p$, reduces to

$$\dot{J}_p = -\sum_{k=1}^{n} V^k_0 \rho^k(t) z^k_a \bar{B}^k, \tag{84}$$

where all n actuators are assumed to be located above the mid-plane. Unlike the anisotropic plate, vibration suppression may now be accomplished by pairing every actuator with a corresponding sensor located anywhere on the opposite side of the mid-plane. From equation (66) it follows that the current generated by the kth corresponding sensor layer located at a distance $-z_s^k$ (z_s^k is defined as positive in sign) with respect to (below) the mid-plane may be expressed as

$$i^k(t) = z_s^k \bar{B}^k. \tag{85}$$

By enforcing the constraint $\rho^k(t) = \text{sgn}\,[i^k(t)]$, the energy flux expression becomes

$$\dot{J}_p = -\sum_{k=1}^{n} V_0^k z_a^k |\bar{B}^k|, \tag{86}$$

which is negative definite for all choices in $\Lambda^k(x, y)$ for which the preceding expression is non-vanishing. Thus for an orthotropic plate in pure bending it is *sufficient* to guarantee global stability if the control input to each actuator layer located anywhere above the neutral mid-plane is any function which is of the same sign as the current generated by an identical spatially distributed sensor located anywhere below the mid-plane. An identical sensor located above the plate mid-plane, and a feedback law the sign of which is always opposite to the sensed current, would be equally valid.

In an entirely analogous manner, if the orthotropic plate experiences pure membrane stretching in the absence of bending and torsion, the resulting (sufficient) design constraint may be even further relaxed: an identically polarized sensor laminate may be located *anywhere* on the plate and a negative feedback law imposed. If one assumes that the kth actuator and sensor layers are identically spatially distributed, then

$$\dot{J}_p = \sum_{k=1}^{n} V_0^k \rho^k(t) \bar{P}^k \qquad \text{and} \qquad i^k(t) = \bar{P}^k \tag{87, 88}$$

regardless of where the sensor layer is located in the composite structure. Thus, if the control law

$$\rho^k(t) = -\text{sgn}\,[i^k(t)] \tag{89}$$

is imposed, then $\dot{J}_p$ will be guaranteed negative semidefinite. Forcing $\dot{J}_\text{p}$ to be always negative in order to ensure active vibration suppression depends upon suitable choices for the spatial weighting functions, $\Lambda^k(x, y)$.

4. CONCLUSIONS

A control design strategy has been developed for the active vibration control of fully anisotropic plates in which the active elements are laminated piezoelectric layers. The control strategy is based on the second method of Lyapunov, where a Lyapunov functional indicative of the total system mechanical energy was chosen. Asymptotic stability of the composite structure was shown to be ensured as long as three easily realizable sufficient conditions are met. *First*, for each piezoelectric actuator laminate above the composite mid-plane there must exist a corresponding identically polarized sensor laminate, also located above the mid-plane. *Second*, a linear control law governing each conjugate sensor/actuator pair must be enforced such that the input to a given actuator is always proportional and opposite in sign to the current induced by the corresponding sensor. *Third*, for each conjugate pair above the mid-plane there exists an identical pair below the mid-plane. When these criteria are satisfied, active vibration control becomes uniquely dependent on the choice of spatial weighting functions, $\Lambda^k(x, y)$. The analysis indicates that previously employed design strategies for the bending vibration control of isotropic beams and plates can be destablizing in the presence of anisotropy. The study shows that these strategies succeed as a subclass of the general piezo-laminated anisotropic plate control theory relevant to orthotropic and isotropic systems, and represent a relaxation of the three sufficient stability criteria stated in the preceding section.

The viability of spatially weighting the control authority has been demonstrated successfully for isotropic beams [3] and plates [14], and represents a broad topic suitable for future analyses. The results of this study suggest that for anisotropic plates, as well, controller effectiveness may be weighted in favor of individual modes, modal subsets or all modes of a system through a proper choice of the spatial weighting functions, $\Lambda^k(x, y)$. The added complexity of the anisotropic problem prevents the straightforward application of symmetry arguments employed by Burke and Hubbard [4, 12], and new approaches must be developed suitable to the general anisotropic plate problem.

Another area for further work may be to expand the plate laminate model to include other dissipative forces in general, and structural damping in particular. The control design has been based on locating the actuator and sensor layers such as to ensure the negative semidefiniteness of $\dot{J}_p$. As asserted previously, this statement is dictated by intuitive insight: $\dot{J}_p \leqslant 0$ should be a sufficient condition for asymptotic stability regardless of boundary conditions due to the many dissipative forces existent in real structures. Experimental and numerical verification of the present control methodology should be undertaken to enhance the theoretical study put forward in this paper.

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