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Nonlinear Observability Analysis of Spacecraft Attitude and Angular Rate with Inertia Uncertainty[1]

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Abstract

The observability of spacecraft attitude and angular rate is analyzed using notions from nonlinear systems theory. Exploiting the distinctive structure of the system, expressions for its Lie derivatives of any order are derived. The observability mapping is then constructed and analyzed, yielding that the system is nonuniformly observable. Necessary and sufficient conditions for the system to become unobservable are derived. It is further shown that the angular rate unobservability conditions, thus derived, are a generalization of previously obtained results. Additionally, the observability of the inertia tensor is examined, and it is shown that the inertia is nonuniformly observable. A sufficient condition is given under which the control input (generated, e.g., by the momentum wheel) renders the inertia unobservable. This phenomenon is numerically demonstrated using a Bayesian grid-based filter that is applied to the estimation of the attitude and angular rate of a spacecraft subjected to inertia tensor uncertainty.

Introduction

Spacecraft attitude is a vital piece of information in any space mission. Usually, the attitude is estimated on-the-fly by means of a filtering algorithm utilizing both body-fixed and inertial-frame vector measurements. These estimation algorithms often rely on some knowledge of the attitude-rate (angular-rate), which, thereby, permits improved sequential attitude filtering.

The source of body-fixed observations can be a star-tracker, Sun sensor, Earth sensor, or a three-axis magnetometer (TAM). Whereas high-accuracy star-trackers

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are extremely expensive and Sun sensors are useless during Sun eclipse (for low Earth orbit satellites), the TAM is an integrated part of virtually any spacecraft, and its readings (albeit not as accurate as the star-tracker's) are readily available.

A widely used angular rate sensor onboard spacecraft is the rate-gyroscope triad, whose purpose is to provide three-axis rate information. Long experience has shown that rate gyros are failure-prone. They tend to saturate during high angular rate scenarios such as tumbling and initial attitude acquisition. Moreover, gyros may not be suitable for lowcost satellites due to price, power consumption, weight and volume considerations. This has led, in the last decade, to the derivation of gyroless attitude/angular-rate estimation schemes, that provide backup capabilities for spacecraft that use rate gyros, and affordable solutions for low-cost gyroless satellites.

Several methods were proposed in the past for combined attitude and angular rate filtering in the absence of rate sensors [1–7]. Without exception, all of these methods use the traditional attitude parameterization used in space applications, the rotation quaternion. The main advantage of using the quaternion representation is that it is not singular for any rotation. Moreover, its kinematic equation is linear and the computation of the associated attitude matrix involves only algebraic expressions.

A rather different class of algorithms has been recently introduced for angular rate estimation from vector observations [8–11]. In this approach, the angular rate is estimated independently of any attitude or orbital information. This class of estimators relies on the underlying assumption that the inertial vector measurement source (i.e., the Sun direction vector or the magnetic field vector) is nearly constant in the inertial frame between two successive measurements.

It is well known, from systems theory, that hidden signals (e.g., the attitude and the angular rate) can be reconstructed only if they are observable. This clearly implies that the filtering performance strongly depends on the system's observability. Therefore, it is of prime interest to determine whether a system is observable or not, and, if it is not unconditionally observable, to identify the conditions under which it loses observability. In the case at hand the attitude and angular rate problem is nonlinear. This fact, in turn, renders the observability analysis nontrivial.

Only a few works have specifically dealt with the observability of the problem at hand. Thus, conclusions regarding the observability of the angular rate are drawn by both references [8] and [9]. Both of these references assume a stationary spacecraft, hence, their findings are not applicable in the dynamic case. An intuitive remark concerning the spacecraft inertia tensor identifiability is stated as part of the derivation of a robust angular rate estimator in reference [10].

This paper aims at providing a rigorous, deterministic, observability analysis of the combined attitude and angular rate estimation problem. The problem of inertia tensor identifiability is analyzed as well. The approach taken is based on using notions from nonlinear systems theory. Exploiting the distinctive structure of the system at hand, its Lie derivatives of any order are analytically derived. This allows constructing the system's unique observability mapping. It is then proven that the system is nonuniformly observable. Finally, necessary and sufficient conditions under which the system is unobservable are derived. The essence of the obtained results is demonstrated using a discrete-state filtering example.

The remainder of this paper is organized as follows. The next section provides some necessary preliminaries from nonlinear systems theory. Then, the unique observability mapping associated with the attitude and angular rate (AAR) system is derived. The next section is concerned with the observability analysis of the AAR system. An illustrative discrete-state filtering example then follows. Concluding remarks are offered in the last section.

Preliminaries: Observability of Nonlinear Systems

Consider a general nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \tag{1b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^p$, and $\mathbf{u} \in \mathcal{U}$, are the system's state, output and admissible control input vectors set in $\mathbb{R}^m$, respectively.

The following definition is adopted from reference [12]:

Definition 1 (Observability). *The system is observable if the knowledge of* $\mathbf{y}$ *on a finite time interval is sufficient to determine the initial state uniquely.*

Assuming that $\mathbf{f}$, $\mathbf{h}$, and $\mathbf{u}$ are C^∞-functions, allows constructing an infinite-dimensional observability mapping consisting of the output $\mathbf{y}$ and its time derivatives (see equation (2) below). If the system is observable then its state is uniquely parameterized via the entries of the observability mapping.

Corresponding to equations (1), the following definitions extend the notions introduced in reference [12] to account for systems driven by a control input.

Definition 2 (Observability Mapping). *The observability mapping of the system in equations* (1) *is defined as*

$$\mathcal{O}(\mathbf{x}(t), \mathbf{u}(t)) \triangleq \begin{bmatrix} \mathbf{y}(t) \\ \dot{\mathbf{y}}(t) \\ \ddot{\mathbf{y}}(t) \\ \vdots \\ \mathbf{y}^{(k)}(t) \\ \vdots \end{bmatrix} \tag{2}$$

Alternatively, the observability mapping can be expressed using the system's state and control-input derivatives, that is

$$\mathcal{O}(\mathbf{x}(t), \mathbf{u}(t)) =$$
$$\begin{bmatrix} \mathcal{L}_f^0 \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathcal{L}_f \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) + \left[\dfrac{\partial \mathbf{y}(t)^{\mathrm{T}}}{\partial \mathbf{u}(t)}\right]^{\mathrm{T}} \dot{\mathbf{u}}(t) \\ \mathcal{L}_f^2 \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) + \mathcal{L}_f\left(\left[\dfrac{\partial \mathbf{y}(t)^{\mathrm{T}}}{\partial \mathbf{u}(t)}\right]^{\mathrm{T}} \dot{\mathbf{u}}(t)\right) + \left[\dfrac{\partial \dot{\mathbf{y}}(t)^{\mathrm{T}}}{\partial \mathbf{u}(t)}\right]^{\mathrm{T}} \dot{\mathbf{u}}(t) + \left[\dfrac{\partial \dot{\mathbf{y}}(t)^{\mathrm{T}}}{\partial \dot{\mathbf{u}}(t)}\right]^{\mathrm{T}} \ddot{\mathbf{u}}(t) \\ \vdots \\ \mathcal{L}_f \mathbf{y}^{(k-1)}(t) + \Sigma_{i=0}^{k-1} \left[\dfrac{\partial \mathbf{y}^{(k-1)}(t)^{\mathrm{T}}}{\partial \mathbf{u}^{(i)}(t)}\right]^{\mathrm{T}} \mathbf{u}^{(i+1)}(t) \\ \vdots \end{bmatrix} \tag{3}$$

where $\mathcal{L}_f^k \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t))$ is the kth-order Lie derivative of the smooth function $\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t))$ with respect to the vector field $\mathbf{f}(\mathbf{x}(t)), \mathbf{u}(\mathbf{t}))$. The Lie derivative of any order is computed via the recursion

$$\mathcal{L}_f^k \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) = \left[\frac{\partial (\mathcal{L}_f^{k-1} \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)))^{\mathrm{T}}}{\partial \mathbf{x}(t)} \right]^{\mathrm{T}} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{4}$$

with

$$\mathcal{L}_f^0 \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \triangleq \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \tag{5}$$

where the matrix on the right-hand side of equation (4) is the Jacobian of the Lie derivative $\mathcal{L}_f^{k-1} \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t))$ with respect to $\mathbf{x}(t)$.

Two initial states $\mathbf{x}_1(0), \mathbf{x}_2(0)$ are distinguishable if their corresponding output signals differ. If this holds for every couple of initial states then the system is said to be globally observable. In order to derive an observability criterion, it is convenient to use the following definition:

Definition 3 (Global Observability Condition). *Let*

$$\mathcal{D}(\mathbf{x}(t), \mathbf{u}(t)) \triangleq \{\mathbf{z}(t) \in \mathbb{R}^n \,|\, \mathcal{O}(\mathbf{x}(t), \mathbf{u}(t)) = \mathcal{O}(\mathbf{z}(t), \mathbf{u}(t))\} \tag{6}$$

be the set of all state space vectors corresponding to the control input $\mathbf{u}(t)$*, that cannot be distinguished from the vector* $\mathbf{x}(t)$*. Then, the system in equations* (1) *is said to be globally observable if* $\mathcal{D}(\mathbf{x}(t), \mathbf{u}(t)) = \{\mathbf{x}(t)\}$ *for all* $\mathbf{x}(t) \in \mathbb{R}^n$ *and for all* $\mathbf{u}(t) \in \mathcal{U}$.

Definition 4 (Nonuniformly Observable Systems). *A nonlinear system is said to be nonuniformly observable if there exists an admissible control input for which it is unobservable.*

Observability Mapping of the AAR System

Determining whether a nonlinear system is observable consists of identifying the state space vectors composing the set $\mathcal{D}(\mathbf{x}(t), \mathbf{u}(t))$. Usually, this task is nontrivial since equation (6) involves the solution of an infinite number of equations. However, in the case under consideration, a distinctive structure of the observability mapping greatly facilitates the analysis.

The AAR System

A generalized continuous-time model of the AAR system is given by the first-order differential equations [5]

$$\dot{\mathbf{q}}(t) = \frac{1}{2} \Psi(\boldsymbol{\omega}(t)) \mathbf{q}(t) \tag{7a}$$

$$\dot{\boldsymbol{\omega}}(t) = -J^{-1} [\dot{\mathbf{H}}(t) + \boldsymbol{\omega}(t) \times (J\boldsymbol{\omega}(t) + \mathbf{H}(t))] \tag{7b}$$

and the measurement equation

$$\mathbf{y}(t) = A(\mathbf{q}(t)) \mathbf{r}(t) \tag{8}$$

where $\mathbf{y}(t)$ and $\mathbf{r}(t)$ denote the body-fixed and the reference-frame observation vectors, respectively. Both observation vectors are related through an attitude matrix $A(\cdot)$ associated with the quaternion of rotation

$$\mathbf{q} \triangleq \begin{bmatrix} \boldsymbol{\rho} \\ q_4 \end{bmatrix} \tag{9}$$

where $\boldsymbol{\rho}$ and q_4 denote the quaternion's vector and scalar parts, respectively. The spacecraft angular rate vector $\boldsymbol{\omega}(t)$ satisfies Euler's dynamics equation (7b) which involves the inertia tensor J and a control input $\mathbf{H}$ (introduced, e.g., by the spacecraft momentum wheel). The quaternion kinematics, equation (7a), involves the angular rate-dependent transition matrix $\Psi(\boldsymbol{\omega}(t))$, defined as

$$\Psi(\boldsymbol{\omega}(t)) \triangleq \begin{bmatrix} -[\boldsymbol{\omega}(t)\times] & \boldsymbol{\omega}(t) \\ -\boldsymbol{\omega}(t)^{\mathrm{T}} & 0 \end{bmatrix} \tag{10}$$

where $[\boldsymbol{\omega}(t)\times]$ denotes the usual cross-product matrix associated with the vector $\boldsymbol{\omega}(t)$. In this system the known control inputs are the reference-frame observation vector $\mathbf{r}(t)$ and the angular momentum $\mathbf{H}(t)$.

Remark 1. Unless otherwise stated, the inertia tensor, J, is assumed to be a diagonal matrix. The rationale behind this is based on the fact that it is always possible to formulate the rigid body dynamics in a reference frame aligned with the principal axes.

In what follows, the explicit time dependency is abandoned for the sake of notational simplicity. Also, for brevity, the following definition is used

$$\Gamma_J^{H}(\boldsymbol{\omega}) \triangleq J^{-1}\,[\dot{\mathbf{H}} + \boldsymbol{\omega} \times (J\boldsymbol{\omega} + \mathbf{H})] \tag{11}$$

Definition (11) allows writing the angular rate dynamics equation as

$$\dot{\boldsymbol{\omega}} = -\Gamma_J^{H}(\boldsymbol{\omega}) \tag{12}$$

The Observability Mapping

Being an infinite dimensional vector, the observability mapping cannot usually be explicitly computed. Fortunately, as far as the AAR system is concerned, this problem can be alleviated. In particular, it turns out that whenever the inertia tensor is formulated in a reference frame aligned with the principal axes, the AAR system's associated Lie derivatives of any order admit a unique generalized form.

Corresponding to the system's governing equations, let the vector field $\mathbf{f}(\mathbf{q}, \boldsymbol{\omega})$ and the output function $\mathbf{h}(\mathbf{q}, \mathbf{r})$ be defined as

$$\mathbf{f}(\mathbf{q}, \boldsymbol{\omega}) \triangleq \left[\frac{1}{2}\,(\Psi(\boldsymbol{\omega})\mathbf{q})^{\mathrm{T}}, -\Gamma_J^{H}(\boldsymbol{\omega})^{\mathrm{T}}\right]^{\mathrm{T}} \tag{13a}$$

$$\mathbf{h}(\mathbf{q}, \mathbf{r}) \triangleq A(\mathbf{q})\mathbf{r} \tag{13b}$$

Then, the kth-order Lie derivative can be computed recursively using the following lemma:

Lemma 1 (System Lie Derivatives). *The kth-order Lie derivative of the function* $\mathbf{h}(\mathbf{q}, \mathbf{r})$ *with respect to the vector field* $\mathbf{f}(\mathbf{q}, \boldsymbol{\omega})$ *takes the form*

$$\mathcal{L}_f^k \mathbf{h}(\mathbf{q}, \mathbf{r}) = \sum_{i=1}^{3^{k-1}} B_i^k\,[\mathbf{h}(\mathbf{q}, \mathbf{r}) \times]\mathbf{v}_i^k,\ k \geq 1 \tag{14}$$

where $\mathbf{v}_i^k \in \mathbb{R}^3$ *and* $B_i^k \in \mathbb{R}^{3\times 3}$ *are independent of* $\mathbf{q}$*, and can be computed recursively as follows. For* $k = 1$*:* $\mathbf{v}_1^1 = \boldsymbol{\omega}$*,* $B_1^1 = I_{3\times 3}$*. For* $k \geq 2$*, the vector* $\mathbf{v}_i^k$ *may take* *any of the values* $\boldsymbol{\omega}$*, or* $\mathbf{v}_j^{k-1}$*, or* $\dot{\mathbf{v}}_j^{k-1}$*, where* $j \in [1, 3^{k-2}]$*. Corresponding to the value of* $\mathbf{v}_i^k$*, the matrix* B_i^k *takes the values* $-B_j^{k-1}[\mathbf{v}_j^{k-1}\times]$*, or* $\dot{B}_j^{k-1}$*, or* B_j^{k-1}*.*

Proof. The lemma is proven using mathematical induction. Thus, it is first shown to hold for $k = 1$.

Using equation (13) in equation (4), it immediately follows that

$$\mathcal{L}_f^k \mathbf{h}(\mathbf{q}, \mathbf{r}) = \frac{1}{2}\left[\frac{\partial(\mathcal{L}_f^{k-1}\mathbf{h}(\mathbf{q}, \mathbf{r}))^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Psi(\boldsymbol{\omega})\mathbf{q} + \left[\frac{\partial(\mathcal{L}_f^{k-1}\mathbf{h}(\mathbf{q}, \mathbf{r}))^{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right]^{\mathrm{T}} \dot{\boldsymbol{\omega}} \tag{15}$$

Therefore

$$\mathcal{L}_f \mathbf{h}(\mathbf{q}, \mathbf{r}) = \frac{1}{2}\left[\frac{\partial \mathbf{h}(\mathbf{q}, \mathbf{r})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Psi(\boldsymbol{\omega})\mathbf{q} \tag{16}$$

Now, let the matrix $\Xi(\mathbf{q})$ be defined as

$$\Xi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\rho}\times] \\ -\boldsymbol{\rho}^{\mathrm{T}} \end{bmatrix} \tag{17}$$

Using some known properties of this matrix [13], the right-hand side of equation (16) can be rearranged, yielding

$$\frac{1}{2}\left[\frac{\partial \mathbf{h}(\mathbf{q}, \mathbf{r})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Psi(\boldsymbol{\omega})\mathbf{q} = \frac{1}{2}\left[\frac{\partial \mathbf{h}(\mathbf{q}, \mathbf{r})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Xi(\mathbf{q})\boldsymbol{\omega} \tag{18}$$

A direct (albeit tedious) computation of the right-hand side of equation (18), finally yields[4]

$$\mathcal{L}_f \mathbf{h}(\mathbf{q}, \mathbf{r}) = \frac{1}{2}\left[\frac{\partial \mathbf{h}(\mathbf{q}, \mathbf{r})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Xi(\mathbf{q})\boldsymbol{\omega} = [\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\boldsymbol{\omega} \tag{19}$$

To prove that the lemma holds for every $k > 1$, assume that it holds for $k - 1$. Then, the Lie derivative $\mathcal{L}_f^{k-1}\mathbf{h}(\mathbf{q}, \mathbf{r})$ can be expressed as

$$\mathcal{L}_f^{k-1}\mathbf{h}(\mathbf{q}, \mathbf{r}) = \sum_{j=1}^{3^{k-2}} B_j^{k-1}[\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\mathbf{v}_j^{k-1} \tag{20}$$

where $B_j^{k-1} \in \mathbb{R}^{3\times 3}$ and $\mathbf{v}_j^{k-1} \in \mathbb{R}^3$ are independent of $\mathbf{q}$. Using equation (20) in equation (15), and using the identity in equation (19), the kth-order Lie derivative is given by

$$\begin{aligned}
\mathcal{L}_f^k \mathbf{h}(\mathbf{q}, \mathbf{r}) &= \frac{1}{2}\sum_{j=1}^{3^{k-2}}\left[\frac{\partial(B_j^{k-1}[\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\mathbf{v}_j^{k-1})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Psi(\boldsymbol{\omega})\mathbf{q} \\
&\quad + \sum_{j=1}^{3^{k-2}}\left[\frac{\partial(B_j^{k-1}[\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\mathbf{v}_j^{k-1})^{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right]^{\mathrm{T}} \dot{\boldsymbol{\omega}} \\
&= -\frac{1}{2}\sum_{j=1}^{3^{k-2}} B_j^{k-1}[\mathbf{v}_j^{k-1}\times]\left[\frac{\partial \mathbf{h}(\mathbf{q}, \mathbf{r})^{\mathrm{T}}}{\partial \mathbf{q}}\right]^{\mathrm{T}} \Psi(\boldsymbol{\omega})\mathbf{q} \\
&\quad + \sum_{j=1}^{3^{k-2}} B_j^{k-1}[\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\left[\frac{\partial(\mathbf{v}_j^{k-1})^{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right]^{\mathrm{T}} \dot{\boldsymbol{\omega}} + \sum_{j=1}^{3^{k-2}}\sum_{l=1}^{3} \dot{\omega}_l \frac{\partial B_j^{k-1}}{\partial \omega_l}[\mathbf{h}(\mathbf{q}, \mathbf{r})\times]\mathbf{v}_j^{k-1}
\end{aligned}$$

[4]A similar result appears in reference [14].

$$= -\sum_{j=1}^{3k-2} B_j^{k-1}[\mathbf{v}_j^{k-1}\times][\mathbf{h}(\mathbf{q},\mathbf{r})\times]\boldsymbol{\omega} + \sum_{j=1}^{3k-2} B_j^{k-1}[\mathbf{h}(\mathbf{q},\mathbf{r})\times]\dot{\mathbf{v}}_j^{k-1}$$
$$+ \sum_{j=1}^{3k-2} \dot{B}_j^{k-1}[\mathbf{h}(\mathbf{q},\mathbf{r})\times]\mathbf{v}_j^{k-1}$$
$$= \sum_{j=1}^{3k-2}\{-B_j^{k-1}[\mathbf{v}_j^{k-1}\times][\mathbf{h}(\mathbf{q},\mathbf{r})\times]\boldsymbol{\omega} + B_j^{k-1}[\mathbf{h}(\mathbf{q},\mathbf{r})\times]\dot{\mathbf{v}}_j^{k-1}$$
$$+ \dot{B}_j^{k-1}[\mathbf{h}(\mathbf{q},\mathbf{r})\times]\mathbf{v}_j^{k-1}\} \tag{21}$$

which can be expressed using the general form (14). □

Lemma 2 (AAR System's Observability Mapping). *The kth entry of the AAR system's observability mapping is given by*

$$\mathbf{y}^{(k)} = \sum_{i=0}^{k} a_i^k \mathcal{L}_f^i \mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-i)}) \tag{22}$$

where $a_i^k = \binom{k}{i}$.

Proof. The lemma is proven using mathematical induction. For $k = 1$, equation (3) gives the first entry in the observability mapping as

$$\dot{\mathbf{y}} = \mathcal{L}_f \mathbf{h}(\mathbf{q},\mathbf{r}) + \left[\frac{\partial \mathbf{y}^{\mathrm{T}}}{\partial \mathbf{r}}\right]^{\mathrm{T}} \dot{\mathbf{r}} = \mathcal{L}_f \mathbf{h}(\mathbf{q},\mathbf{r}) + \mathbf{h}(\mathbf{q},\dot{\mathbf{r}}) \tag{23}$$

Assume now that the lemma holds for $k-1$, that is

$$\mathbf{y}^{(k-1)} = \sum_{i=0}^{k-1} a_i^{k-1} \mathcal{L}_f^i \mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-1-i)}) \tag{24}$$

with $a_i^{k-1} = \binom{k-1}{i}$. Then, equation (3) yields

$$\mathbf{y}^{(k)} = \mathcal{L}_f \mathbf{y}^{(k-1)} + \sum_{i=0}^{k-1}\left[\frac{\partial(\mathbf{y}^{(k-1)})^{\mathrm{T}}}{\partial \mathbf{r}^{(i)}}\right]^{\mathrm{T}} \mathbf{r}^{(i+1)} = \sum_{i=1}^{k} a_{i-1}^{k-1} \mathcal{L}_f^i \mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-i)})$$
$$+ \sum_{i=0}^{k-1}\left[\frac{\partial(\mathbf{y}^{(k-1)})^{\mathrm{T}}}{\partial \mathbf{r}^{(i)}}\right]^{\mathrm{T}} \mathbf{r}^{(i+1)} \tag{25}$$

From equations (14) and (24) it follows that the last term on the right-hand side of equation (25) satisfies

$$\sum_{i=0}^{k-1}\left[\frac{\partial(\mathbf{y}^{(k-1)})^{\mathrm{T}}}{\partial \mathbf{r}^{(i)}}\right]^{\mathrm{T}} \mathbf{r}^{(i+1)} = \sum_{i=0}^{k-1} a_{k-1-i}^{k-1} \mathcal{L}_f^{k-1-i} \mathbf{h}(\mathbf{q}, \mathbf{r}^{(i+1)}) = \sum_{j=0}^{k-1} a_j^{k-1} \mathcal{L}_f^j \mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-1-j+1)}) \tag{26}$$

The lemma follows upon substituting equation (26) into equation (25), and using Pascal's rule for the binomial coefficients. □

Observability Analysis

This section is concerned with analyzing the observability of the attitude and angular rate estimation problem. Thus, several conditions are given under which either the attitude or the angular rate are unobservable. The main insights are discussed and compared with the previously obtained results of references [8–10].

The following definition is used in the sequel.

Definition 5. *A vector* $\mathbf{a}$ *is said to be stationary if it is nonrotating, that is* $\mathbf{a}/\|\mathbf{a}\| = const$.

Lemma 3. *If and only if* $\mathbf{a}$ *is stationary then*

$$\mathbf{a}^{(i)} = c_{i-1}\mathbf{a}^{(i-1)} = \cdots = c_1\dot{\mathbf{a}} = c_0\mathbf{a} \tag{27}$$

where $c_i \in \mathbb{R}$ *are possibly time-dependent.*

Proof. Since $\dot{\mathbf{a}} = c_0/c_1\mathbf{a}$, it immediately follows that the ith differential equation in equation (27) is equivalent to

$$\dot{\mathbf{a}} = c\mathbf{a} \tag{28}$$

Let $\bar{\mathbf{a}} \triangleq \mathbf{a}/\|\mathbf{a}\|$, then

$$\dot{\bar{\mathbf{a}}} = \frac{\dot{\mathbf{a}}}{\|\mathbf{a}\|} - \frac{\mathbf{a}}{\|\mathbf{a}\|^3}\mathbf{a}^{\mathrm{T}}\dot{\mathbf{a}} \tag{29}$$

Substituting equation (28) into equation (29) yields

$$\dot{\bar{\mathbf{a}}} = 0 \tag{30}$$

implying that $\boldsymbol{a}$ is stationary. The converse direction follows easily from equation (29). □

Attitude Observability

Theorem 1. *The attitude quaternion is exclusively unobservable (i.e., the two states* $[\mathbf{q}'^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$ *and* $[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$, *where* $\mathbf{q}' \neq \mathbf{q}$, *are indistinguishable) if and only if the reference-frame vector is stationary. In that case the set of indistinguishable states is*

$$\mathcal{D}_q^* = \left\{[\mathbf{q}'^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}} | \mathbf{q}' = \mathbf{e}\otimes\mathbf{q}, \quad \mathbf{e} = \left[\mathbf{y}^{\mathrm{T}}/\|\mathbf{y}\|\sin\frac{\theta}{2}, \quad \cos\frac{\theta}{2}\right]^{\mathrm{T}}, \quad \theta\in[0, 2\pi]\right\} \tag{31}$$

for all $\boldsymbol{\omega} \in \mathbb{R}^3$.

Proof. In order to prove the attitude observability condition, both Lemmas 1 and 2 are used. Recognizing that the only expressions which depend exclusively upon the quaternion in the kth entry of the observability mapping are $\mathbf{h}(\mathbf{q}, \mathbf{r}^{(i)})$, $i = 0, \ldots, k$, it immediately follows that two states comprising different attitudes $\mathbf{q}' \neq \mathbf{q}$ are indistinguishable if and only if

$$\mathbf{h}(\mathbf{q}', \mathbf{r}^{(i)}) = A(\mathbf{q}')\mathbf{r}^{(i)} = A(\mathbf{q})\mathbf{r}^{(i)} = \mathbf{h}(\mathbf{q}, \mathbf{r}^{(i)}), \qquad i = 0, \ldots, \infty \tag{32}$$

Since the attitude is completely specified by at least two noncollinear reference vectors, it follows that equation (32) is satisfied if and only if

$$\mathbf{r}^{(i)} = c_{i-1}\mathbf{r}^{(i-1)} = \cdots = c_1\dot{\mathbf{r}} = c_0\mathbf{r} \tag{33}$$

where $c_i \in \mathbb{R}$ are possibly time-dependent, implying that $\mathbf{r}$ is stationary (see Lemma 3). The set of indistinguishable quaternions are then obtained as the solution of

$$A(\mathbf{q}')\mathbf{r} = A(\mathbf{q})\mathbf{r} \tag{34}$$

Letting $\mathbf{q}' = \mathbf{e} \otimes \mathbf{q}$, for some quaternion $\mathbf{e}$, and manipulating equation (34), yields

$$[A(\mathbf{e}) - I_{3\times3}]\, A(\mathbf{q})\mathbf{r} = \mathbf{0} \tag{35}$$

where the identity $A(\mathbf{e} \otimes \mathbf{q}) = A(\mathbf{e})A(\mathbf{q})$ was used. Equation (35) implies that $\mathbf{y} = A(\mathbf{q})\mathbf{r}$ is an eigenvector of $A(\mathbf{e})$ corresponding to the eigenvalue $\lambda = 1$, which consequently means that $\mathbf{y}$ is aligned along the rotation axis of $\mathbf{e}$. Therefore, the quaternion $\mathbf{e}$ can be expressed as

$$\mathbf{e} = \left[\mathbf{y}^{\mathrm{T}} / \| \mathbf{y} \| \sin\frac{\theta}{2}, \quad \cos\frac{\theta}{2} \right]^{\mathrm{T}}, \quad \theta \in [0, 2\pi] \tag{36}$$

yielding the set in equation (31). □

Remark 2. From Theorem 1 it follows that the AAR system is non-uniformly observable.

Angular Rate Observability

Theorem 2. *The angular rate becomes exclusively unobservable (i.e., the two states* $[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$ *and* $[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}'^{\mathrm{T}}]^{\mathrm{T}}$*, where* $\omega \neq \omega'$*, are indistinguishable) if and only if*

$$\boldsymbol{\omega} \times \mathbf{h}(\mathbf{q}, \mathbf{r}^{(i)}) = \mathbf{0} \tag{37a}$$

and

$$\dot{\boldsymbol{\omega}} \times \mathbf{h}(\mathbf{q}, \mathbf{r}^{(i)}) = \mathbf{0} \tag{37b}$$

for all $i = 0, \ldots, \infty$.

Proof. (If) Observing both Lemmas 1 and 2, the kth entry of the observability map is written as

$$\mathbf{y}^{(k)} = \mathbf{h}(\mathbf{q}, \mathbf{r}^{(k)}) + \sum_{i=1}^{k} a_i^k \sum_{j=1}^{3i-1} B_j^i [\mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-i)}) \times] \mathbf{v}_j^i \tag{38}$$

which implies that whenever

$$\mathbf{h}(\mathbf{q}, \mathbf{r}^{(k-i)}) \times \mathbf{v}_j^i = \mathbf{0}, \qquad \forall\, i, j, k \tag{39}$$

the angular rate becomes unobservable. Also, Lemma 1 shows that, by nature of their recursive computation, the functions $\mathbf{v}_j^i$ can be expressed as time derivatives of the angular rate, that is

$$\mathbf{v}_j^i = \boldsymbol{\omega}^{(n)} \tag{40}$$

for some $n < i$.

Now, if equations (37) hold, both the angular rate and the angular acceleration vectors are collinear, implying that the angular rate is stationary. This renders all vectors $\mathbf{v}_j^i$ in equation (40) collinear and aligned along the angular rate vector, which in turn means that equation (39), the condition for angular rate unobservability, is satisfied because of equation (37a).

(Only if) The converse direction is proven by contradiction. Let $\mathcal{D}^*$ be the set of all states comprising indistinguishable angular rate parts, and suppose that there exist two distinct states $[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D}^*$ and $[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}'^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D}^*$ with $\boldsymbol{\omega}' \neq \boldsymbol{\omega}$ that do not satisfy either of conditions (37).

Explicitly writing the first and second terms in the observability mapping using equations (38) and (21) yields

$$\dot{\mathbf{y}} = \mathbf{h}(\mathbf{q}, \dot{\mathbf{r}}) + [\mathbf{h}(\mathbf{q}, \mathbf{r}) \times]\boldsymbol{\omega} \tag{41a}$$

$$\ddot{\mathbf{y}} = \mathbf{h}(\mathbf{q}, \ddot{\mathbf{r}}) + 2\,[\mathbf{h}(\mathbf{q}, \dot{\mathbf{r}}) \times]\boldsymbol{\omega} - [\boldsymbol{\omega}\times][\mathbf{h}(\mathbf{q}, \mathbf{r}) \times]\boldsymbol{\omega} + [\mathbf{h}(\mathbf{q}, \mathbf{r}) \times]\dot{\boldsymbol{\omega}} \tag{41b}$$

Because the condition in equation (37a) is assumed not to hold, and since the two states differ, the cross product operator in equation (41a) renders the system's observability mapping distinctive (because $\dot{\mathbf{y}} \neq \dot{\mathbf{y}}'$), implying that both states are distinguishable. This is a contradiction to the fact that both states belong to $\mathcal{D}^*$, so equation (37a) must hold for angular rate unobservability.

Assume now that equation (37a) holds, but equation (37b) does not. Then, equation (41b) yields

$$\ddot{\mathbf{y}} = \mathbf{h}(\mathbf{q}, \ddot{\mathbf{r}}) + [\mathbf{h}(\mathbf{q}, \mathbf{r}) \times]\dot{\boldsymbol{\omega}} \tag{42}$$

Now, equation (37a) yields $\boldsymbol{\omega} = g\mathbf{h}(\mathbf{q}, \mathbf{r})$ and $\boldsymbol{\omega}' = g'\mathbf{h}(\mathbf{q}, \mathbf{r})$ for some possibly time-varying functions g and g' such that $g \neq g'$, hence

$$\dot{\boldsymbol{\omega}} = -\Gamma_J^{\mathrm{H}}(\boldsymbol{\omega}) \neq \dot{\boldsymbol{\omega}}' = -\Gamma_J^{\mathrm{H}}(\boldsymbol{\omega}') \tag{43}$$

implying that $\ddot{\mathbf{y}} \neq \ddot{\mathbf{y}}'$. This, in turn, renders the two states distinguishable, thus contradicting the fact that both belong to $\mathcal{D}^*$. □

Theorem 2 gives rise to the following insights.

Corollary 1. *The angular rate unobservability conditions in Theorem 2 are satisfied if and only if both of the following conditions hold: 1) both the reference-frame vector r and the angular rate are stationary, and 2) both the angular rate and the body-fixed measurement vectors are collinear. In this case the set of indistinguishable states is*

$$\mathcal{D}_\omega^* = \{[\mathbf{q}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}} \mid \mathbf{q} \in \mathbb{S}^3, \quad \boldsymbol{\omega} = c\mathbf{h}(\mathbf{q}, \mathbf{r}), \quad c \in \mathbb{R}\} \tag{44}$$

Proof. Equations (37) are satisfied if and only if

$$\mathbf{r}^{(i)} = c_{i-1}\,\mathbf{r}^{(i-1)} = \cdots = c_1\,\dot{\mathbf{r}} = c_0\,\mathbf{r}, \qquad c_i \in \mathbb{R} \tag{45a}$$

and

$$\dot{\boldsymbol{\omega}} = a\boldsymbol{\omega}, \qquad a \in \mathbb{R} \tag{45b}$$

where a and c_i are possibly time-dependent. Based on Lemma 3, equations (45) imply that both $\mathbf{r}$ and $\boldsymbol{\omega}$ are stationary. The set of indistinguishable states is then obtained from equation (37a). □

Remark 3. Obviously, Corollary 1 holds also if the angular rate stationarity condition is replaced by the stricter condition of a steady state angular motion, that is, $\dot{\boldsymbol{\omega}} = \mathbf{0}$.

Corollary 1 enables computing a control input function that can render the angular rate unobservable, as stated in the following corollary.

Corollary 2. *A control input satisfying*

$$\dot{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H} = -\boldsymbol{\omega} \times J\boldsymbol{\omega} + cJ\boldsymbol{\omega} \tag{46}$$

for some possibly time-varying function c renders the angular rate unobservable if the reference-frame vector is stationary and both the angular rate and the body-fixed measurement vectors are collinear.

Proof. It is easy to verify that the momentum control input satisfying equation (46) renders the angular rate vector stationary (i.e., $\dot{\boldsymbol{\omega}} = c\boldsymbol{\omega}$). The corollary then follows from Corollary 1. □

Discussion. Conveyed by Theorem 2 and Corollary 1, a situation which partially guarantees an unobservable angular rate occurs whenever the reference-frame vector is stationary, and both the angular-rate and the body-fixed measurement vectors are collinear. Bearing this in mind, and writing down the time-derivative of the body-fixed measurement vector $\mathbf{y} = \mathbf{h}(\mathbf{q}, \mathbf{r})$ in this case, yields

$$\dot{\mathbf{y}} = \frac{\mathrm{d}A(\mathbf{q})}{\mathrm{d}t}\mathbf{r} + A(\mathbf{q})\dot{\mathbf{r}} = -\boldsymbol{\omega} \times \mathbf{y} + c\mathbf{y} = c\mathbf{y} \tag{47}$$

implying, by Lemma 3, that $\mathbf{y}$ is stationary as well. This result states that the reference-frame vector is stationary and both the angular rate and the body-fixed measurement vectors are collinear if and only if the body-fixed measurement is stationary. Therefore, recalling Corollary 1, the following is concluded:

Corollary 3. *The angular rate is exclusively unobservable if and only if both the angular rate and the body-fixed measurement vectors are stationary.*

Another insight that concurs with intuition is inferred from both Theorem 1 and Corollary 1. Thus, it can be recognized that an unobservable angular rate imposes an unobservable attitude. However, the converse direction is not necessarily true.

Reduced-order angular rate system. There are situations in which one is interested in estimating the angular rate independently of the attitude [8–11]. In such cases, the corresponding system model is given as a reduced-order version of equation (7) as

$$\dot{\boldsymbol{\omega}} = -\Gamma_J^{\mathrm{H}}(\boldsymbol{\omega}) \tag{48}$$

The corresponding measurement equation is obtained by assuming that the reference-frame vector rate of change, $\dot{\mathbf{r}}$, is negligible in equation (47), yielding an effective measurement equation as

$$\mathbf{z} = -\boldsymbol{\omega} \times \mathbf{y} \tag{49}$$

where $\mathbf{z} \triangleq \dot{\mathbf{y}}$. Notice that in this system, the measurements are both the body-fixed vector and its rate of change.

The following theorem states necessary and sufficient conditions for the above system to be unobservable.

Theorem 3. *The reduced-order angular rate system is unobservable if and only if*

$$\boldsymbol{\omega} \times \mathbf{y} = \mathbf{0} \tag{50a}$$

and

$$\dot{\boldsymbol{\omega}} \times \mathbf{y} = \mathbf{0} \tag{50b}$$

In that case the set of indistinguishable states is

$$\mathcal{D}^* = \{\boldsymbol{\omega} \mid \boldsymbol{\omega} = c\mathbf{y}, \quad c \in \mathbb{R}\} \tag{51}$$

Proof. (If) Using equation (50a), the observability mapping of the reduced-order system is derived as

$$\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \\ \vdots \\ \mathbf{z}^{(k)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \times \dot{\boldsymbol{\omega}} \\ \mathbf{y} \times \ddot{\boldsymbol{\omega}} \\ \vdots \\ \mathbf{y} \times \boldsymbol{\omega}^{(k)} \\ \vdots \end{bmatrix} \tag{52}$$

Now, equation (50b) yields $\dot{\boldsymbol{\omega}} = a\boldsymbol{\omega}$, where $a \in \mathbb{R}$ is possibly time-dependent. Because now $\boldsymbol{\omega}^{(k)} = a^k \boldsymbol{\omega}$, the observability mapping (52) vanishes, rendering the two states $\boldsymbol{\omega} = c\mathbf{y}$ and $\boldsymbol{\omega}' = c'\,\mathbf{y}$, for the possibly time-varying functions $c \neq c'$, indistinguishable. The latter observation yields the set $\mathcal{D}^*$.

(Only if) The converse direction is proven by contradiction. Let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{D}^*$, and suppose that $\boldsymbol{\omega} \neq \boldsymbol{\omega}'$. If equation (50a) does not hold, then the cross product operator in equation (49) renders the system's observability mapping distinctive (because $\mathbf{z} = \mathbf{y} \times \boldsymbol{\omega} \neq \mathbf{y} \times \boldsymbol{\omega}' = \mathbf{z}'$), implying that both states are distinguishable, thus contradicting the fact that both belong to $\mathcal{D}^*$. If equation (50a) holds but equation (50b) does not, then from equation (50a) it follows that $\boldsymbol{\omega} = c\mathbf{y}$ and $\boldsymbol{\omega}' = c'\mathbf{y}$, for some possibly time-varying functions c and c' such that $c \neq c'$, yielding

$$\dot{\boldsymbol{\omega}} = -\Gamma_J^{\mathrm{H}}(\boldsymbol{\omega}) \neq \dot{\boldsymbol{\omega}}' = -\Gamma_J^{\mathrm{H}}(\boldsymbol{\omega}') \tag{53}$$

Equation (53) implies that $\dot{\mathbf{z}} \neq \dot{\mathbf{z}}'$, which in turn renders the two states distinguishable, contradicting the assumption that both belong to $\mathcal{D}^*$. □

Corollary 4. *The reduced-order angular rate system is unobservable if and only if both the angular rate and the body-fixed measurement vectors are stationary.*

Proof. Observing equation (49) it is concluded that the body-fixed measurement vector is stationary if and only if equation (50a) is satisfied. Thus, alternatively, the first condition in equation (50a) may be replaced by the prerequisite of a stationary body-fixed vector. Assuming that the first condition does hold, the second condition (50b) is equivalent to $\dot{\boldsymbol{\omega}} = a\boldsymbol{\omega}$, for some possibly time-varying function $a \in \mathbb{R}$, which holds if and only if $\boldsymbol{\omega}$ is a stationary vector (see Lemma 3). □

It should be noted that, unsurprisingly, both Corollaries 3 and 4 are identical. This concurs with the fact that the angular-rate system is a reduced-order version of the complete AAR system.

Inertia Tensor Observability

As was pointed out previously, the sensitivity of Euler's equation to inertia modeling imperfections gave rise to several angular-rate estimation algorithms that inherently cope with such uncertainties [9–11]. In these algorithms, the inertia tensor entries are estimated rather than being regarded as exact. Therefore, the question of whether the inertia tensor is observable is of prime importance in such cases.

Assuming unvarying inertia, it is further deduced that the observability mappings of both the complete and the reduced-order systems, augmented by the inertia tensor entries, remain unchanged. Therefore, the inertia tensor appears exclusively in the terms of $\boldsymbol{\omega}^{(k)}$ in equations (52) and (40). The latter observation implies that whenever Euler's equation (or the angular acceleration) is invariant to inertia change the system is unobservable. This fact is intuitively inferred assuming no angular momentum control input (i.e., $\dot{\mathbf{H}} = \mathbf{H} = \mathbf{0}$) in reference [10], leading to the estimation of only two entries of the complete inertia tensor.

In general one could think of a control input $\mathbf{H}$ rendering Euler's equation invariant to inertia changes. A sufficient condition identifying such a control input is stated in the following proposition.

Proposition 1. *A control input satisfying*

$$\dot{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H} = [J^{-1} - (J')^{-1}]^{-1} [(J')^{-1}(\boldsymbol{\omega} \times J'\boldsymbol{\omega}) - J^{-1}(\boldsymbol{\omega} \times J\boldsymbol{\omega})] \tag{54}$$

for some inertia tensors J and J' renders these two inertias indistinguishable.

Proof. Substituting equation (54) into equation (48) and assuming the spacecraft inertia is either J or J', yields, after rearranging,

$$\dot{\boldsymbol{\omega}} = -(J' - J)^{-1} [\boldsymbol{\omega} \times (J' - J)\boldsymbol{\omega}] \tag{55}$$

in both cases. Consequently, J and J' are indistinguishable. □

Relation to Previously Obtained Results

It turns out that the angular-rate observability conditions obtained previously in both references [8] and [9] are special cases of Corollary 4. In both works the spacecraft is assumed to be in steady-state condition (i.e., $\dot{\boldsymbol{\omega}} = \mathbf{0}$), which consequently means that equation (50b) is automatically satisfied. For completeness, the exact claims of both references are revisited.

Reference [8] states that the angular rate is unobservable whenever $\boldsymbol{\omega}$, the angular-momentum, and the body-fixed observation vector are collinear. These conditions imply

$$\dot{\boldsymbol{\omega}} = -J^{-1} (\boldsymbol{\omega} \times J\boldsymbol{\omega}) = \mathbf{0} \tag{56a}$$

and

$$\boldsymbol{\omega} \times \mathbf{y} = \mathbf{0} \tag{56b}$$

Equations (56) are a special case of the conditions stated by Theorem 3.

Reference [9] states that the angular rate is unobservable whenever the body-fixed observation vector is collinear with both a principal axis of inertia, and the spacecraft angular momentum, that is

$$J\mathbf{y} = \lambda\mathbf{y} \tag{57a}$$

and

$$\mathbf{y} \times J\boldsymbol{\omega} = \mathbf{0} \tag{57b}$$

Equations (57) imply that $\boldsymbol{\omega}$ is collinear with $\mathbf{y}$ (i.e., $\boldsymbol{\omega} = c\mathbf{y}$), thus

$$\dot{\boldsymbol{\omega}} = -J^{-1} (\boldsymbol{\omega} \times J\boldsymbol{\omega}) = -c^2 J^{-1} (\mathbf{y} \times J\mathbf{y}) = -\lambda c^2 J^{-1} (\mathbf{y} \times \mathbf{y}) = \mathbf{0} \tag{58}$$

In other words, in both references [8] and [9] the angular rate vector is assumed to be time invariant, yielding exclusively the condition of stationary body-fixed vector (i.e., equation (50a)). Theorem 3, on the other hand, provides an additional condition corresponding to the dynamic case in which the angular rate is time varying.

Another implication, following naturally from Theorem 3, is that during an instantaneous situation in which a stationary body-fixed vector is coupled by a rotating angular rate vector (i.e., just equation (50a) is satisfied) the angular rate is observable. Moreover, in contrary to previous results, Theorem 3 shows the necessity of the obtained conditions as well.

Discrete-State Filtering Example

The essence of the preceding results is demonstrated using a simplified discrete-time filtering example. In particular, the AAR system's inertia tensor is shown to be nonuniformly observable as a Bayesian grid-based filter is implemented for estimating the joint posterior probability mass function (pmf) of the attitude and the angular rate under inertia uncertainty.

Discrete State AAR System

Suppose that the sample space (i.e., the set of all possible outcomes) of the AAR system's state augmented by the three diagonal inertia elements is a random set consisting of a finite number of entities. Let $\mathcal{Q}_k = \{\mathbf{q}_k(i)\}_{i=1}^{Nq}$, $\mathcal{R}_k = \{\Omega_k(i)\}_{i=1}^{N\omega}$ and $\mathcal{J} = \{[J_{11}(i), J_{22}(i), J_{33}(i)]^{\mathrm{T}}\}_{i=1}^{\mathrm{N_J}}$ be the sets of possible attitude quaternions, angular-rate and inertia discrete states at time t_k, respectively. Thus, the complete sample space at time t_k, $\mathcal{S}_k$, is the Cartesian product of $\mathcal{Q}_k$, $\mathcal{R}_k$ and $\mathcal{J}$, that is

$$\mathcal{S}_k = \mathcal{Q}_k \times \mathcal{R}_k \times \mathcal{J} \tag{59}$$

where the number of elements composing the set $\mathcal{S}_k$ is $N = N_q N_\omega N_J$. Assuming that the vector measurements are acquired at evenly spaced time instants t_1, t_2, $t_3, \ldots$, and are contaminated by a zero-mean white Gaussian noise, the observation equation at time t_k takes the form

$$\mathbf{y}_k = A(\mathbf{q}_k)\mathbf{r}_k + \mathbf{n}_k \tag{60}$$

which, in turn, yields the likelihood probability density function

$$p_{\mathbf{y}_k|\mathbf{q}_k}(\mathbf{Y}_k \mid \mathbf{q}_k) = p_{\mathbf{n}_k}(\mathbf{Y}_k - A(\mathbf{q}_k)\mathbf{r}_k) \tag{61}$$

where $\mathbf{n}_k$ and $\mathbf{Y}_k$ denote the measurement noise random vector and the measurement realization at time t_k, respectively. Furthermore, the elements of the initial set $\mathcal{S}_o$ are known to be equally probable. This fact is mathematically expressed as

$$p_{\mathbf{s}_0}(\mathbf{S}_0(i)) = \Pr(\mathbf{s}_0 = \mathbf{S}_0(i)) = \frac{1}{N}, \qquad i = 1, \ldots, N \tag{62}$$

where $\Pr(\mathbf{s}_0 = \mathbf{S}_0(i))$ denotes the probability of the event $\mathbf{s}_0 = \mathbf{S}_0(i)$, and

$$\mathbf{S}_k(i) \triangleq [\mathbf{q}_k^{\mathrm{T}}(i), \mathbf{\Omega}_k^{\mathrm{T}}(i), [J_{11}(i), J_{22}(i), J_{33}(i)]^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{S}_k \qquad i = 1, \ldots, N \tag{63}$$

For simplicity, it is further assumed that the AAR system's dynamical motion is purely deterministic (i.e., there is no process noise).

Now, suppose that one is interested in sequentially estimating the system's state using the vector observations. Considering the case at hand, it is convenient to implement an exact Bayesian grid-based method for that purpose [15].

Bayesian Grid-Based Filter

Let $s^k \triangleq \{\mathbf{s}_0, \ldots, \mathbf{s}_k\}$ and $\mathcal{Y}^k \triangleq \{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ be the system's state and observations time histories up to time k, respectively, and let $S^k \triangleq \{\mathbf{S}_0, \ldots, \mathbf{S}_k\}$ and

$\mathcal{Y}^k \triangleq \{\mathbf{Y}_1, \ldots, \mathbf{Y}_k\}$ be the realizations of s^k and $\mathcal{Y}^k$, respectively. Recalling that the dynamical motion is deterministic, the transition pmf $p_{\mathbf{q}_k,\boldsymbol{\omega}_k|\mathbf{q}_{k-1},\boldsymbol{\omega}_{k-1},J}$ is simply

$$p_{\mathbf{q}_k,\boldsymbol{\omega}_k|\mathbf{q}_{k-1},\boldsymbol{\omega}_{k-1},J}(\mathbf{q}_k, \boldsymbol{\Omega}_k|\mathbf{q}_{k-1},\boldsymbol{\Omega}_{k-1}, J) = \Pr([\mathbf{q}_k^{\mathrm{T}}, \boldsymbol{\Omega}_k^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{f}_{\mathbf{H}}(\mathbf{q}_{k-1}, \boldsymbol{\Omega}_{k-1}, J)) \tag{64}$$

where $\mathbf{f}_{\mathbf{H}}(\mathbf{q}_{k-1}, \boldsymbol{\Omega}_{k-1}, J)$ represents the solution at time t_k of the system in equation (7) with initial conditions $\mathbf{q}(t_{k-1}) = \mathbf{q}_{k-1}$, $\boldsymbol{\omega}(t_{k-1}) = \boldsymbol{\omega}_{k-1}$, inertia tensor J and some known control-input $\mathbf{H}$. Because J is constant, it follows that

$$p_{\mathbf{s}_k|\mathbf{s}_{k-1}}(\mathbf{S}_k \,|\, \mathbf{S}_{k-1}) = p_{\mathbf{q}_k,\boldsymbol{\omega}_k|\mathbf{q}_{k-1},\boldsymbol{\omega}_{k-1},J}(\mathbf{q}_k, \boldsymbol{\Omega}_k \,|\, \mathbf{q}_{k-1}, \boldsymbol{\Omega}_{k-1}, J) \tag{65}$$

Using both equations (61) and (65) while recalling that given the state at time k, $\mathbf{y}_k$ is statistically independent of its past, $\mathcal{Y}^{k-1}$, the joint pmf of the measurements and the state time histories is obtained as

$$\begin{aligned} p_{s^k,\mathcal{Y}^k}(S^k, Y^k) &= p_{\mathbf{s}_0}(S_0)\prod_{i=1}^{k} p_{\mathbf{y}_i|\mathbf{s}_i}(\mathbf{Y}_i \,|\, \mathbf{S}_i) p_{\mathbf{s}_i|\mathbf{s}_{i-1}}(\mathbf{S}_i \,|\, \mathbf{S}_{i-1}) \\ &= p_{\mathbf{s}_0}(S_0)\prod_{i=1}^{k} p_{\mathbf{y}_i|\mathbf{q}_i}(\mathbf{Y}_i \,|\, \mathbf{q}_i) p_{\mathbf{s}_i|\mathbf{s}_{i-1}}(\mathbf{S}_i \,|\, \mathbf{S}_{i-1}) \end{aligned} \tag{66}$$

where the identity $p_{\mathbf{y}_i|\mathbf{s}_i} = p_{\mathbf{y}_i|\mathbf{q}_i}$ was used. A recursive form of equation (66) is given by

$$p_{s^k,\mathcal{Y}^k}(S^k, Y^k) = p_{\mathbf{y}_k|\mathbf{q}_k}(\mathbf{Y}_k \,|\, \mathbf{q}_k) p_{\mathbf{s}_k|\mathbf{s}_{k-1}}(\mathbf{S}_k \,|\, \mathbf{S}_{k-1}) p_{s^{k-1},\mathcal{Y}^{k-1}}(S^{k-1}, Y^{k-1}) \tag{67}$$

Because the joint pmf $p_{\mathbf{s}^k,\mathcal{Y}^k}$ is proportional to $p_{\mathbf{s}^k|\mathcal{Y}^k}$, the maximum a posteriori (MAP) estimate of the state trajectory $\hat{S}^k$, is obtained as

$$\hat{S}^k = S^K(i^*) \tag{68a}$$

where

$$i^* = \arg\max_{i\in[1,N]} p_{\mathbf{s}^k,\mathcal{Y}^k}(S^k(i), Y^k) \tag{68b}$$

and $S^k(i)$ denotes the ith discrete state trajectory. Furthermore, letting

$$J(i) \triangleq \operatorname{diag}\{J_{11}(i), J_{22}(i), J_{33}(i)\}, \qquad i \in [1, N_J] \tag{69}$$

and recalling that $\Pr(J = J(i)) = 1/N_J$, $i = 1, \ldots, N_J$, the likelihood of $J(i)$ given the measurements up to time k is computed as

$$p_{\mathcal{Y}^k|J}(Y^k \,|\, J(i)) \propto p_{J,\mathcal{Y}^k}(J(i), Y^k) = \sum_{j=1}^{N} p_{\mathbf{s}^k,\mathcal{Y}^k}(S^k(j), Y^k) \Pr(J_{S^k(j)} = J(i)) \tag{70}$$

where $J_{S^k(j)}$ denotes the inertia tensor associated with the state trajectory $S^k(j)$.

The posterior pmf in equation (66) and the likelihood in equation (70) can be utilized for providing adequate state and inertia estimates as long as the system is observable. In other cases, both equations may be used to identify those states which are indistinguishable. It can be expected that as such, the indistinguishable states should be assigned equal probability masses. The latter claim is numerically assessed by applying both equations (68) and (70), when subjected to a control-input rendering the inertia tensor unobservable.

Example

The filtering example considers sets of $N_q = 10$ quaternions, $N_{\boldsymbol{\omega}} = 10$ angular-rate samples, and $N_J = 4$ inertia tensors. All initial sets are drawn from uniform distributions, as follows: each of the first three (vector) quaternion components is sampled from $U[-0.5, 0.5]$, whereas the additional element is set to satisfy the unit norm constraint. Each of the three components of the angular-rate vector is sampled from $U[-0.5, 0.5]$ rad/sec. Finally, the three inertia elements are drawn from

$$J_{nn}(i) \sim U[200, 500] \text{ kg m}^2, \qquad n = 1, 2, 3, \qquad i = 1, \ldots, N_J \tag{71}$$

In all runs, the measurement noise n_k in equation (60) is sampled from a Gaussian distribution with standard deviation $3I_{3\times3}$ (nT), and the reference-frame vectors are generated using the 8th-order international geomagnetic reference field model, with orbital parameters identical to those used in reference [16].

Two scenarios are examined and compared. In the first scenario, a control input obtained as the solution of equation (54) is applied, thereby rendering the inertia tensor unobservable. The second case, in which the inertia is fully observable, assumes $\mathbf{H} = \text{const}$.

As was pointed out by Proposition 1, injecting a control input that satisfies

$$\begin{aligned}\dot{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H} = [J(i)^{-1} - J(j)^{-1}]^{-1}\,[J(j)^{-1}\,(\boldsymbol{\omega} \times J(j)\boldsymbol{\omega}) \\ - J(i)^{-1}\,(\boldsymbol{\omega} \times J(i)\boldsymbol{\omega})], i, j \in [1, N_J]\end{aligned} \tag{72}$$

where the system's true inertia tensor is either $J(i)$ or $J(j)$, renders both inertias indistinguishable. Thus, in the first scenario, the momentum wheel control input is taken as the solution of equation (72) with $i = 1, j = 4$ (the true inertia tensor is chosen as $J(1)$), and initial condition $\mathbf{H}(0) = \mathbf{0}$. The time histories of such a control input and the corresponding angular rate components during a single run are demonstrated in Fig. 1.

The Bayesian grid-based filter estimation performance is assessed based on 100 Monte Carlo runs. The initial set S_0 is resampled at the beginning of each run, and the system's true initial state is taken as the first element of this set. The MAP norm estimation error distributions of both the quaternion and the angular rate are shown in Fig. 2 using the 95th, 85th, 50th, and 15th percentile curves. From Fig. 2 it can be seen that in 95 percent of the runs the estimation error in both channels vanishes after approximately 35 time steps. Figure 3 shows the average likelihood probability masses, based on 100 runs, of each of the inertia tensors $J(i)$, $i = 1, 2, 3, 4$ (i.e.,

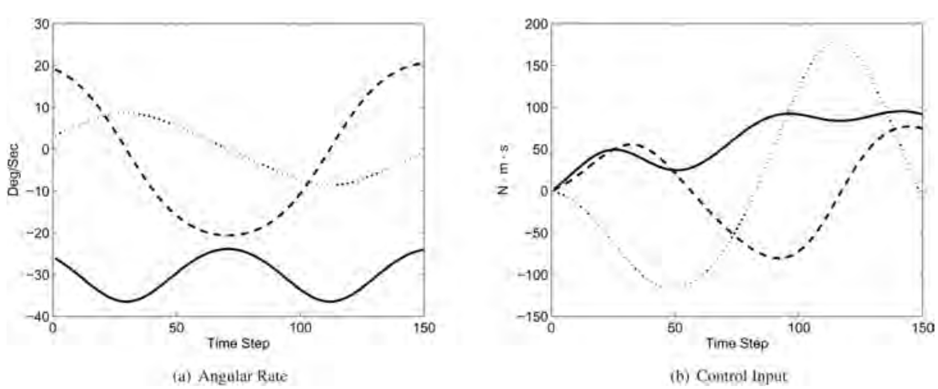


FIG. 1. Left panel: spacecraft angular-rate vector components, ω_1 (solid line), ω_2 (dashed line), ω_3 (dotted line). Right panel: momentum wheel control-input components, H_1 (solid line), H_2 (dashed line), H_3 (dotted line). Single run.

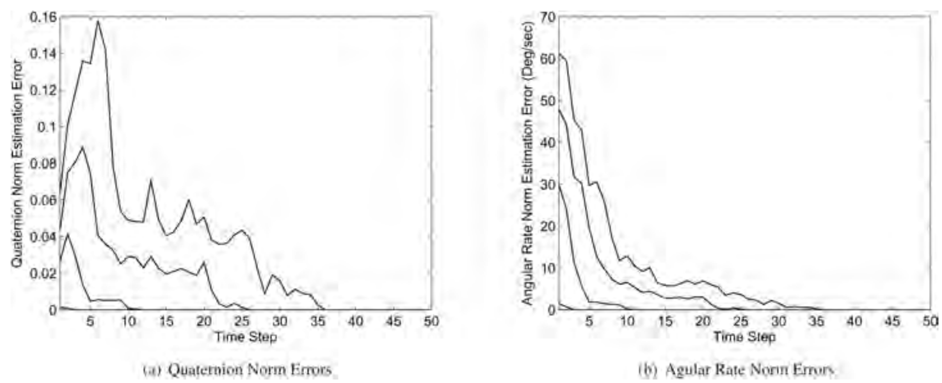


FIG. 2. Distribution of the attitude and angular rate norm estimation errors. Lines, top to bottom: 95th, 85th, 50th and 15th percentiles, computed based on 100 Monte Carlo runs. Unobservable inertia case.

the average of equation (70)). Clearly, the likelihoods of both inertias, $J(1)$ and $J(4)$, reach a similar value of 0.5, reflecting the fact that both states are indistinguishable.

For comparison, the preceding scenario is repeated with the only difference of a constant momentum control input (which is arbitrarily set to $\mathbf{H}(t) = [0.1, 0.2, 0.3]^T$, $\forall\, t \geq 0$, in all runs). Figures 4 and 5 show the corresponding MAP norm estimation error distributions and the inertias likelihood probability masses for this case,

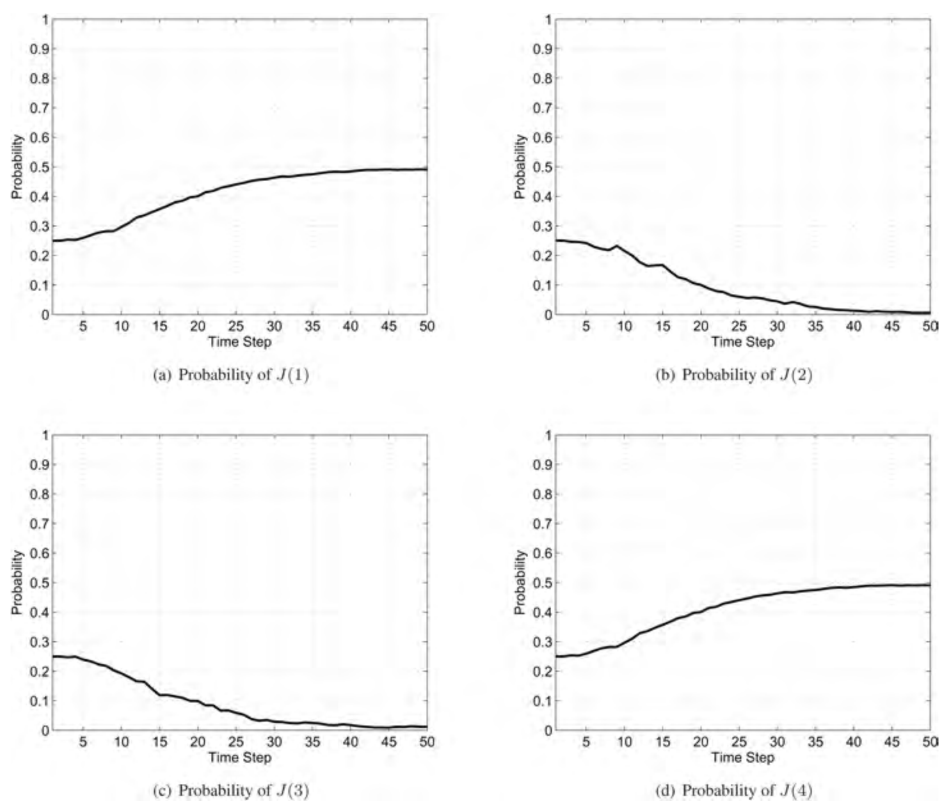
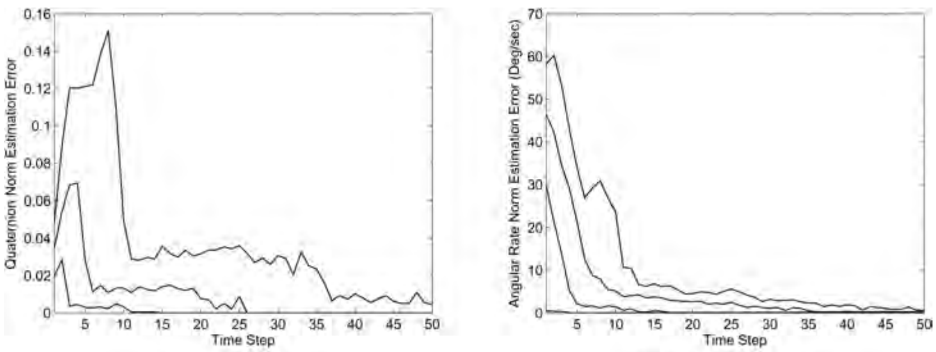


FIG. 3. Average Likelihood Probability Masses of the Four Possible Inertia Tensors. Unobservable Inertia Case.

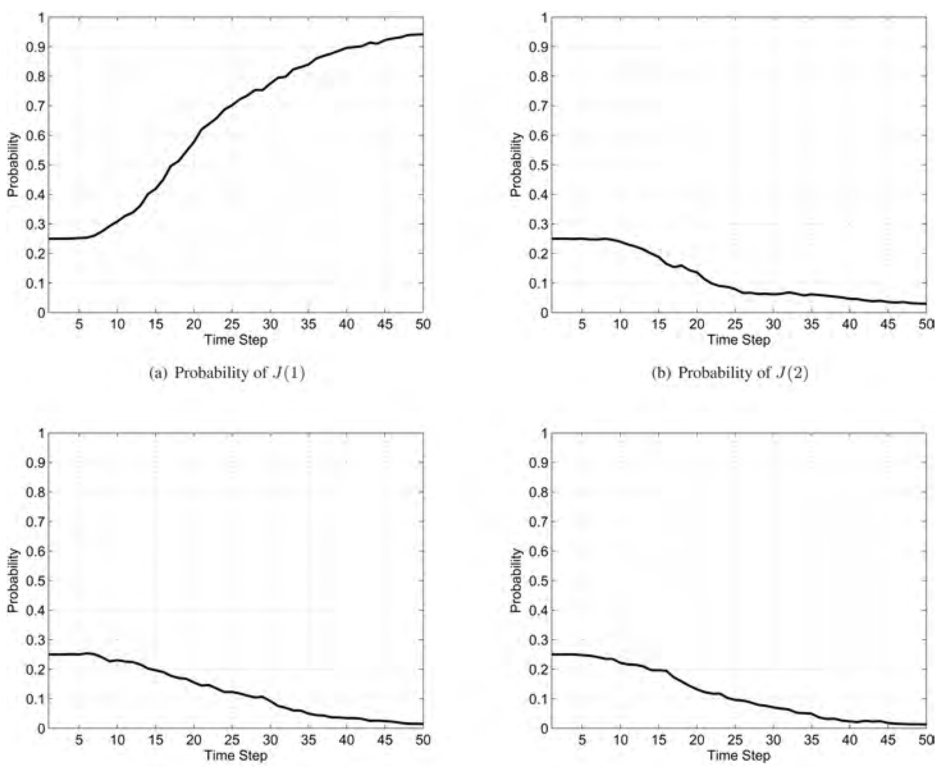


(a) Quaternion Norm Errors

(b) Agular Rate Norm Errors

FIG. 4. Distribution of the attitude and angular rate norm estimation errors. Lines, top to bottom: 95th, 85th, 50th and 15th percentiles, computed based on 100 Monte Carlo runs. Observable inertia case.

respectively. As before, the grid-based filter identifies the correct attitude and angular-rate trajectories in 95 percent of the runs, in which the estimation error vanishes at approximately 47 time steps. In this case, however, the inertia tensor turns out to be fully observable, as the average probability mass of the true inertia $J(1)$ reaches a value of approximately 0.95.



(a) Probability of $J(1)$

(b) Probability of $J(2)$

(c) Probability of $J(3)$

(d) Probability of $J(4)$

FIG. 5. Average Likelihood Probability Masses of the Four Possible Inertia Tensors. Observable Inertia Case.

Conclusions

The observability of the combined attitude and angular rate estimation problem subjected to inertia tensor uncertainties is analyzed. Necessary and sufficient conditions under which this system is unobservable are derived, based on the concept of observability mapping. The conditions, thus derived, render the system nonuniformly observable. The obtained angular rate unobservability conditions are shown to generalize previous results of related works. Finally, the observability of the inertia tensor is examined. It is shown that the inertia is nonuniformly observable. This fact is demonstrated using a discrete-state filtering example.

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