Attitude Determination from Vector Observations: Quaternion Estimation

I.Y. BAR-ITZHACK, Member, IEEE
Y. OSHMAN
Technion—Israel Institute of Technology

Two recursive estimation algorithms, which use pairs of measured vectors to yield minimum variance estimates of the quaternion of rotation, are presented. The nonlinear relations between the direction cosine matrix and the quaternion are linearized, and a variant of the extended Kalman filter is used to estimate the difference between the quaternion and its estimate. With each measurement this estimate is updated and added to the whole quaternion estimate. This operation constitutes a full state reset in the estimation process. Filter tuning is needed to obtain a converging filter. The second algorithm presented uses the normality property of the quaternion of rotation to obtain, in a straightforward design, a filter which converges, with a smaller error, to a normal quaternion. This algorithm changes the state but not the covariance computation of the original algorithm and implies only a partial reset. Results of Monte-Carlo simulation runs are presented which demonstrate the superiority of the normalized quaternion.

Manuscript received April 10, 1984.

Authors' address: Department of Aeronautical Engineering, Technion, Technion City, Haifa 32000, Israel.

I. INTRODUCTION

The problem of determining the attitude of a vehicle by using pairs of vector measurements has been treated by several researchers in the past, using various approaches. Most of the researchers attacked this problem by trying to evaluate the direction cosine matrix (DCM), which is the transformation matrix between some reference coordinate system and the system whose attitude is to be determined, or by trying to evaluate a corresponding sequence of Euler angles. The DCM evaluation was carried out using either a purely deterministic approach [1, 2], or a batch least square fitting [3, 4], or a recursive least square fitting [5]. An algorithm for a recursive minimum variance DCM identification was also recently introduced [6].

Another popular way to represent attitude between two coordinate systems is the quaternion of rotation [7] (or attitude quaternion). Only a few investigators selected the quaternion to determine the attitude from vector measurements. Shuster [8] as well as Shuster and Oh [2] discuss a batch weighted least square algorithm to obtain the quaternion.

A rather general discussion of the application of Kalman filtering to quaternion estimation was recently presented in a survey paper [9] in which the possibility of covariance matrix singularity was given considerable attention. Gai et al. [10] handle a problem similar to the one dealt with in this paper. Their solution yields a different algorithm, which stems from the difference between the two approaches. In particular, they define the difference between the true and the estimated quaternion as a quaternion too; therefore the error is multiplicative in nature, while in our work the difference is a column matrix and is additive.

A recursive minimum variance estimator of the quaternion of rotation which is based on vector measurement is introduced here. The normality property of the quaternion of rotation is used to improve convergence and accuracy. Although this work is concerned with the estimation of only the quaternion of rotation, it can be easily modified, using standard procedures, to estimate other variables such as gyro drifts, misalignment angles, etc. The present work is a natural evolution of the work described by Shuster [8] and Shuster and Oh [2] and was inspired by that work as well as by the parallel work on DCM identification which is described in [6].

The problem attacked in this work is stated in Section II. In Section III the development of a certain quaternion estimation algorithm is presented. Section IV introduces an improved algorithm in which normalization is employed. Section V presents results of Monte-Carlo simulation runs of both algorithms, and the conclusions are discussed in Section VI.

II. PROBLEM STATEMENT

A sequence of vectors r_i, $i = 1,2, \ldots, N$, is measured in two Cartesian coordinate systems u and v. System v is attached to a rotating vehicle and system u is a reference coordinate system. The measurements in system u result in the sequence $\{\boldsymbol{u}_i\}$ $i = 1,2, \ldots, N$ of column matrices, whereas in system v, the measurements result in the corresponding sequence $\{\boldsymbol{v}_i\}$ $i = 1,2, \ldots, N$. The column matrices $\boldsymbol{u}_i$ and $\boldsymbol{v}_i \in R^3$. We wish to compute $\hat{\boldsymbol{q}}$, the minimum variance estimate of $\boldsymbol{q}$, where the latter is a column matrix whose elements are the components of the quaternion of rotation. (In the ensuing we refer to $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ as the quaternion and its estimate, although in reality $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ are column matrices whose elements are those of the quaternion and its estimate, respectively.)

III. QUATERNION ESTIMATION

The Static Case

Unlike the DCM, the relations between the quaternion of rotation q and the column matrices $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are nonlinear. Therefore, in contrast to the DCM identification technique of [6], the whole quaternion cannot be estimated using the ordinary Kalman filter. Rather, a filter similar to the extended Kalman filter (EKF) is used to estimate the difference between the actual quaternion and its estimate. Each newly updated estimate of this difference will be added to the quaternion estimate to form the newly updated (or current) whole quaternion estimate. As a first step in the algorithm development, we derive, in the ensuing, the linear relations between $\delta\boldsymbol{q}$, $\boldsymbol{u}_{i+1}$, and $\boldsymbol{v}_{i+1}$, where $\delta\boldsymbol{q}$ is the difference between $\boldsymbol{q}$ and its estimate $\hat{\boldsymbol{q}}$.

It is well known that D can be expressed in terms of $\boldsymbol{q}$ as follows [11]:

$$D(\boldsymbol{q}) =$$

$$\begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3+q_1q_4) & 2(q_2q_4-q_1q_3) \\ 2(q_2q_3-q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_4q_3+q_1q_2) \\ 2(q_2q_4+q_1q_3) & 2(q_4q_3-q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{bmatrix} \tag{1}$$

where q_j, $j = 1,2,3,4$, are the four components of $\boldsymbol{q}$. Suppose $\boldsymbol{q}$ and hence $D(\boldsymbol{q})$ are known; we may use a first-order Taylor series expansion to compute $D(\boldsymbol{q}+\delta\boldsymbol{q})$ as follows:

$$D(\boldsymbol{q}+\delta\boldsymbol{q}) \sim D(\boldsymbol{q}) + \sum_{j=1}^{4} \left.\frac{\partial D}{\partial q_j}\right|_{\boldsymbol{q}} \delta q_j. \tag{2}$$

Denote

$$A_j(\boldsymbol{q}) = \left.\frac{\partial D}{\partial q_j}\right|_{\boldsymbol{q}}, \qquad j = 1,2,3,4 \tag{3}$$

then the partial differentiation of (1) yields

$$A_1(\boldsymbol{q}) = 2\begin{bmatrix} q_1 & q_4 & -q_3 \\ -q_4 & q_1 & q_2 \\ q_3 & -q_2 & q_1 \end{bmatrix} \tag{4a}$$

$$A_2(\boldsymbol{q}) = 2\begin{bmatrix} q_2 & q_3 & q_4 \\ q_3 & -q_2 & q_1 \\ q_4 & -q_1 & -q_2 \end{bmatrix} \tag{4b}$$

$$A_3(\boldsymbol{q}) = 2\begin{bmatrix} -q_3 & q_2 & -q_1 \\ q_2 & q_3 & q_4 \\ q_1 & q_4 & -q_3 \end{bmatrix} \tag{4c}$$

$$A_4(\boldsymbol{q}) = 2\begin{bmatrix} -q_4 & q_1 & q_2 \\ -q_1 & -q_4 & q_3 \\ q_2 & q_3 & q_4 \end{bmatrix} \tag{4d}$$

Suppose $\hat{\boldsymbol{q}}_{i+1/i}$, the estimate of $\boldsymbol{q}$ at step $i+1$ given the previous i measurements, is known and a new pair of measured column matrices, $\boldsymbol{u}_{i+1}$ and $\boldsymbol{v}_{i+1}$ is obtained. The new error-free measurements satisfy

$$\boldsymbol{v}_{0,i+1} = D(\boldsymbol{q}_{i+1})\, \boldsymbol{u}_{0,i+1}. \tag{5}$$

It is assumed that the measurements $\boldsymbol{u}_{i+1}$ and $\boldsymbol{v}_{i+1}$ of $\boldsymbol{u}_{0,i+1}$ and $\boldsymbol{v}_{0,i+1}$ are contaminated by the noises $\boldsymbol{n}_{u,i+1}$ and $\boldsymbol{n}_{v,i+1}$, respectively, such that

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_{0,i+1} + \boldsymbol{n}_{u,i+1} \tag{6a}$$

$$\boldsymbol{v}_{i+1} = \boldsymbol{v}_{0,i+1} + \boldsymbol{n}_{v,i+1}. \tag{6b}$$

It is further assumed that the noises $\boldsymbol{n}_{u,i+1}$ and $\boldsymbol{n}_{v,i+1}$ are zero mean and white, whose respective covariance matrices are denoted by $R_{u,i+1}$ and $R_{v,i+1}$. Let us express $\boldsymbol{q}_{i+1}$ as

$$\boldsymbol{q}_{i+1} = \hat{\boldsymbol{q}}_{i+1/i} + \delta\boldsymbol{q}_{i+1} \tag{7}$$

which is an implied definition of $\delta\boldsymbol{q}_{i+1}$. Substitution of (6) and (7) into (5) yields

$$\boldsymbol{v}_{i+1} = D(\hat{\boldsymbol{q}}_{i+1/i} + \delta\boldsymbol{q}_{i+1})\,(\boldsymbol{u}_{i+1} - \boldsymbol{n}_{u,i+1}) + \boldsymbol{n}_{v,i+1}. \tag{8}$$

Use can now be made of (2) and (3) to express (8) as follows

$$\begin{aligned} \boldsymbol{v}_{i+1} &- D(\hat{\boldsymbol{q}}_{i+1/i})\, \boldsymbol{u}_{i+1} \\ &= [\sum_{j=1}^{4} A_j(\hat{\boldsymbol{q}}_{i+1/i})\, \delta q_{i+1,j}]\boldsymbol{u}_{i+1} \\ &\quad - [\sum_{j=1}^{4} A_j(\hat{\boldsymbol{q}}_{i+1/i})\, \delta q_{i+1,j}]\, \boldsymbol{n}_{u,i+1} \\ &\quad - D(\hat{\boldsymbol{q}}_{i+1/i})\, \boldsymbol{n}_{u,i+1} + \boldsymbol{n}_{v,i+1}. \end{aligned} \tag{9}$$

The second term on the right-hand side of (9) is a second-order term which can be omitted. The first term on this side can be written as

$$[\sum_{j=1}^{4} A_j(\hat{\boldsymbol{q}}_{i+1/i})\, \delta q_{i+1,j}]\, \boldsymbol{u}_{i+1} = H_{i+1/i}(\hat{\boldsymbol{q}}_{i+1/i}, \boldsymbol{u}_{i+1})\, \delta\boldsymbol{q}_{i+1} \tag{10}$$

where

$$H_{i+1/i} = [\boldsymbol{h}_1 \quad \boldsymbol{h}_2 \quad \boldsymbol{h}_3 \quad \boldsymbol{h}_4] \tag{11a}$$

$$\boldsymbol{h}_j = A_j(\hat{\boldsymbol{q}}_{i+1/1})\, \boldsymbol{u}_{i+1}, \qquad j = 1,2,3,4. \tag{11b}$$

and where

$$\delta \boldsymbol{q}_{i+1} = \begin{bmatrix} \delta q_{i+1,1} \\ \delta q_{i+1,2} \\ \delta q_{i+1,3} \\ \delta q_{i+1,4} \end{bmatrix} \tag{12}$$

Define

$$\boldsymbol{n}_{i+1/i} \triangleq \boldsymbol{n}_{v,i+1} - \hat{D}_{i+1/i}\, \boldsymbol{n}_{u,i+1} \tag{13}$$

where

$$\hat{D}_{i+1/i} \triangleq \hat{D}(\boldsymbol{q}_{i+1/i}) \tag{14}$$

then obviously, $\boldsymbol{n}_{i+1/i}$ is a zero-mean white-noise column matrix whose covariance matrix, $R_{i+1/i}$, is given by

$$R_{i+1/i} \triangleq \text{Cov}\,\{\boldsymbol{n}_{i+1/i}\}$$

$$= R_{v,i+1} + \hat{D}_{i+1/i}\, R_{u,i+1}\, \hat{D}^{\mathrm{T}}_{i+1/i}. \tag{15}$$

Finally, define $\boldsymbol{e}_{i+1}$ as follows

$$\boldsymbol{e}_{i+1} = \boldsymbol{v}_{i+1} - \hat{D}_{i+1/i}\, \boldsymbol{u}_{i+1} \tag{16}$$

then, using (10), (13), and (16), (9) can be written as

$$\boldsymbol{e}_{i+1} = H_{i+1/i}\, \delta \boldsymbol{q}_{i+1} + \boldsymbol{n}_{i+1/i}. \tag{17}$$

Equation (17) is the sought linear relationship between the data column matrix $\boldsymbol{e}_{i+1}$ and the unknown column matrix, $\delta \boldsymbol{q}_{i+1}$, which is the difference between the quaternion of rotation and its estimate (see 7 and 12). Note that $\boldsymbol{e}_{i+1}$, $H_{i+1/i}$ and even $\boldsymbol{n}_{i+1/i}$ are all functions of the current estimate of the quaternion of rotation which is a main feature of the EKF.

The estimation of $\boldsymbol{q}$ is done as follows. Given a new set of measured column matrices $\boldsymbol{u}_{i+1}$, $\boldsymbol{v}_{i+1}$, use (17) as the observation equation in an EKF-like algorithm to obtain a new estimate of $\delta \boldsymbol{q}_{i+1}$, which is denoted by $\delta \hat{\boldsymbol{q}}_{i+1/i+1}$. Then add $\delta \hat{\boldsymbol{q}}_{i+1/i+1}$ to the latest estimate of the quaternion, which is denoted by $\hat{\boldsymbol{q}}_{i+1/i}$, to form $\hat{\boldsymbol{q}}_{i+1/i+1}$, the updated quaternion estimate. The fact that $\delta \hat{\boldsymbol{q}}_{i+1/i+1}$ is added to $\hat{\boldsymbol{q}}_{i+1/i}$ in its entirety, implies an impulsive reset [12], consequently, $\delta \hat{\boldsymbol{q}}_{i+1/i+1}$ has to be zeroed before its propagation in time.

In conclusion, the quaternion estimation algorithm in the static case is as follows.

Between Measurements:

$$\hat{\boldsymbol{q}}_{i+1/i} = \hat{\boldsymbol{q}}_{i/i} \tag{18a}$$

$$P_{i+1/i} = P_{i/i}. \tag{18b}$$

Across Measurements:
State update

$$K_{i+1} = P_{i+1/i} H^{\mathrm{T}}_{i+1/i} (H_{i+1/i} P_{i+1/i} H^{\mathrm{T}}_{i+1/i} + R_{i+1/i})^{-1} \tag{18c}$$

$$\delta \hat{\boldsymbol{q}}_{i+1/i+1} = K_{i+1}\, \boldsymbol{e}_{i+1}. \tag{18d}$$

Estimated quaternion reset

$$\hat{\boldsymbol{q}}_{i+1/i+1} = \hat{\boldsymbol{q}}_{i+1/i} + \delta \hat{\boldsymbol{q}}_{i+1/i+1}. \tag{18e}$$

Covariance update

$$P_{i+1/i+1} = (I - K_{i+1} H_{i+1/i+1})\, P_{i+1/i} \times (I - K_{i+1} H_{i+1/i+1})^{\mathrm{T}} + K_{i+1} R_{i+1/i+1} K^{\mathrm{T}}_{i+1}. \tag{18f}$$

Note that $H_{i+1/i+1}$, as well as $R_{i+1/i+1}$, in (18f) are recalculated using the current estimate $\boldsymbol{q}_{i+1/i+1}$ which was computed in (18e). This step is not a part of the classical EKF algorithm ([13], Table 6.1-1). In fact it is realized from (16) that the data $\boldsymbol{e}_{i+1}$, which the algorithm of (18) processes, is a function of the current estimate of the quaternion. This feature by itself is a divergence from the ordinary EKF. Therefore the algorithm presented in (18) is not the traditional EKF. It was derived using the same rationale which was used in the derivation of the EKF, only it was tailored to the problem on hand and justified in simulations. It was found that the use of the current estimate of $\boldsymbol{q}$ in (18f) was imperative. The use of $H_{i+1/i}$ rather than $H_{i+1/i+1}$ brought about a quick decrease in the value of $\boldsymbol{P}$. That, in turn, reduced the value of K_{i+1} too early, such that the filter rejected new measurements much before the estimation error was properly reduced. This phenomenon almost always caused the filter to diverge.

The Dynamic Case

Coordinate system v usually rotates with respect to coordinate system u. Denote the angular rate vector of this rotation by $\overline{\omega}$. Obviously the quaternion of rotation and hence the column matrix $\boldsymbol{q}$ change from one vector measurement to another. It is well known that the rate of change of $\boldsymbol{q}$ is given by

$$\dot{\boldsymbol{q}} = \Omega\, \boldsymbol{q} \tag{19}$$

where

$$\Omega = \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{20}$$

and whose elements are the three components of $\overline{\omega}$ when coordinates are made in the v system. The true quaternion propagates according to (19); however, the estimated quaternion is propagated computationally according to

$$\dot{\hat{\boldsymbol{q}}} = \hat{\Omega}\, \hat{\boldsymbol{q}} \tag{21}$$

where $\tilde{\Omega}$ is a matrix similar to Ω, except that its entries are $\tilde{\omega}_x$, $\tilde{\omega}_y$, and $\tilde{\omega}_z$ which are the three measured angular rates. Since the gyros, which supply these three rate components, are not ideal, the measurements are contaminated with noise; hence

$$\bar{\tilde{\omega}} = \tilde{\omega} + \delta\tilde{\omega} \tag{22}$$

where $\delta\tilde{\omega}$ is the vector noise component and $\bar{\tilde{\omega}}$ is the measured angular rate vector. Following (22), (19) can be written as

$$\dot{\boldsymbol{q}} = (\tilde{\Omega} - \delta\Omega)\,\boldsymbol{q} \tag{23}$$

in which $\delta\Omega$ is a matrix similar to Ω. The elements of $\delta\Omega$ are the components of $\delta\tilde{\omega}$, when the latter is resolved in system v; that is, the components of $\delta\tilde{\omega}$, and hence of $\delta\Omega$, are the three gyro error components. Equation (23) can be written as

$$\boldsymbol{q} = \tilde{\Omega}\,\boldsymbol{q} + B\,\delta\boldsymbol{\omega} \tag{24}$$

where

$$B = \frac{1}{2}\begin{bmatrix} q_2 & q_3 & q_4 \\ -q_1 & q_4 & -q_3 \\ -q_4 & -q_1 & q_2 \\ q_3 & -q_2 & -q_1 \end{bmatrix} \tag{25}$$

and $\delta\boldsymbol{\omega}$ is a column matrix whose elements are the three gyro error components. Now when (21) is subtracted from (24), the following dynamics equation is obtained for $\delta\boldsymbol{q}$

$$\delta\dot{\boldsymbol{q}} = \tilde{\Omega}\,\delta\boldsymbol{q} + B\,\delta\boldsymbol{\omega}. \tag{26}$$

When (26) is discretized, the following difference equation is obtained for the propagation of $\delta\boldsymbol{q}$

$$\delta\boldsymbol{q}_{i+1} = \phi_i\,\delta\boldsymbol{q}_i + B_i\,\delta\omega_i. \tag{27}$$

Note that ϕ_i is a function of the measured angular rate vector $\bar{\tilde{\omega}}$ and B_i is a function of $\boldsymbol{q}_i$. Since $\boldsymbol{q}_i$ itself is not known we use its estimate $\hat{\boldsymbol{q}}_i$ to compute B_i. The latter is a well known characteristic of the EKF.

In our work it is assumed that $\delta\boldsymbol{\omega}_i$ is a zero-mean white-noise column matrix, although other typical gyro noise models could be easily used instead (see [13], pp. 78–84). We do this in order to focus the attention to the estimation techniques of the quaternion itself with the knowledge that the inclusion of typical gyro and misalignment models is straightforward. Following (27), $\delta\hat{\boldsymbol{q}}$ is propagated between measurements according to

$$\delta\hat{\boldsymbol{q}}_{i+1/i} = \phi_i\,\delta\hat{\boldsymbol{q}}_{i/i}. \tag{28}$$

However, like in the static case, an implied full reset is manifested in the recomputation of $\hat{\boldsymbol{q}}$ after each update of $\delta\hat{\boldsymbol{q}}$. This drives $\delta\hat{\boldsymbol{q}}_{i/i}$, and consequently $\delta\hat{\boldsymbol{q}}_{i+1/i}$, to zero.

The estimation error covariance matrix is propagated according to

$$P_{i+1/i} = \phi_i\,P_{i/i}\,\phi_i^{\mathrm{T}} + B_i\,Q_i\,B_i^{\mathrm{T}} \tag{29}$$

where $Q_i = \mathrm{cov}\{\delta\boldsymbol{\omega}_i\}$. In addition, it is concluded from (21) that

$$\hat{\boldsymbol{q}}_{i+1/i} = \phi_i\,\hat{\boldsymbol{q}}_{i/i}. \tag{30}$$

Equations (28)–(30) can now be used to modify (18). This yields the following algorithm for the dynamic case.

Between Measurements:

$$\hat{q}_{i+1/i} = \phi_i\,\hat{q}_{i/i} \tag{31a}$$

$$P_{i+1/i} = \phi_i\,P_{i/i}\,\phi_i^{\mathrm{T}} + B_i\,Q_i\,B_i^{\mathrm{T}}. \tag{31b}$$

Across Measurements:
State update

$$K_{i+1} = P_{i+1/i}\,H_{i+1/i}^{\mathrm{T}}\,(H_{i+1/i}\,P_{i+1/i}\,H_{i+1/i}^{\mathrm{T}} + R_{i+1/i})^{-1} \tag{31c}$$

$$\delta\hat{\boldsymbol{q}}_{i+1/i+1} = K_{i+1}\,\boldsymbol{e}_{i+1}. \tag{31d}$$

Estimated quaternion reset

$$\hat{\boldsymbol{q}}_{i+1/i+1} = \hat{\boldsymbol{q}}_{i+1/i} + \delta\hat{\boldsymbol{q}}_{i+1/i+1}. \tag{31e}$$

Covariance update

$$P_{i+1/i+1} = (I - K_{i+1}\,H_{i+1/i+1})\,P_{i+1/i} \times (I - K_{i+1}\,H_{i+1/i+1})^{\mathrm{T}} + K_{i+1}\,R_{i+1/i+1}\,K_{i+1}^{\mathrm{T}}. \tag{31f}$$

It is shown in Section V that although this filter initially converges very fast, the estimation error reaches a minimum after which it starts diverging slowly. This phenomenon calls for filter tuning [14]. Filter tuning which was carried out by artificial adjustment of the term $B_iQ_iB_i^{\mathrm{T}}$ in (31b) does indeed solve the problem, and a stable filter is obtained. Although an initial value for $B_iQ_iB_i^{\mathrm{T}}$ can be computed, the tuning operation requires as usual a trial and error design mode. In order to eliminate this design mode and obtain in a straightforward manner lower steady state estimation errors, as well as a normal quaternion estimate, normalization is performed in the estimation process. This idea is presented in the next section.

IV. NORMALIZED QUATERNION ESTIMATION

A proper quaternion of rotation possesses the following quality

$$\boldsymbol{q}^{\mathrm{T}}\,\boldsymbol{q} = 1. \tag{32}$$

This relation can be used to upgrade the preceding algorithms. To achieve a lower estimation error and avoid trial and error tuning, the estimated quaternion undergoes optimum normalization each time it is reset; that is, after each application of (18e) or (31e). It has been shown [15] that an optimal normalization in the Euclidean sense is achieved when the quaternion which is to be normalized is simply divided by its norm. Consequently, (18e) and (31e) are followed by

$$\hat{\boldsymbol{q}}^*_{i+1/i+1} = \hat{\boldsymbol{q}}_{i+1/i+1}/\|\,\hat{\boldsymbol{q}}_{i+1/i+1}\|. \tag{33}$$

Now the propagated estimate is $\hat{q}^*_{i+1/i+1}$ rather than $\hat{q}_{i+1/i+1}$, hence (18a) and (31a) have to be replaced by

$$\hat{q}_{i+1/i} = \hat{q}^*_{i/i} \tag{34}$$

and

$$\hat{q}_{i+1/i} = \phi_i\, \hat{q}^*_{i/i} \tag{35}$$

respectively.

The introduction of the normalization operation into the unbiased minimum variance estimation algorithm is an outside intervention in the process. Therefore the effect of this imposed operation has to be examined, and if necessary, steps taken to restore the unbiased minimum variance feature of the algorithm. To accomplish this we examine the implications of the normalization process. Recall that the combination of either (18e) or (31e) and (33) yields the following computation of the normalized estimate

$$\hat{q}^*_{i+1/i+1} = (\hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1})\|\hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1}\|^{-1}. \tag{36}$$

A first-order Taylor series expansion which uses the fact that $\hat{q}_{i+1/i}$ is normal yields

$$\|\hat{q}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}\|^{-1} \sim 1 - \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}. \tag{37}$$

Substitution of (37) into (36) results in

$$\hat{q}^*_{i+1/i+1} \sim (\hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1}) \times (1 - \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}). \tag{38}$$

Neglecting second-order terms, (38) becomes

$$\hat{q}^*_{i+1/i+1} \sim \hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1} - \hat{q}_{i+1/i}\, \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}. \tag{39}$$

The reason we concluded that an impulsive reset was implied in the preceding algorithms stemmed from the fact that the whole estimated quantity $\delta\hat{q}_{i+1/i+1}$ was added to $\hat{q}_{i+1/i}$. Examination of the right-hand side of (39) reveals that to a first-order approximation not all of $\delta\hat{q}_{i+1/i+1}$ is added to $\hat{q}_{i+1/i+1}$ to form the estimate of q_{i+1}, which is then propagated in time, but rather the quantity $\delta\hat{q}_{i+1/i+1}$ less $\hat{q}_{i+1/i}\, \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}$ is added to $\hat{q}_{i+1/i}$. That is, the quantity $\hat{q}_{i+1/i}\, \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1}$ remains as an unreset part of $\delta\hat{q}_{i+1/i+1}$ and has to be propagated between measurements; that is,

$$\delta\hat{q}_{i+2/i+1} = \phi_{i+1}\, \hat{q}_{i+1/i}\, \hat{q}^{\mathrm{T}}_{i+1/i}\, \delta\hat{q}_{i+1/i+1} \tag{40}$$

and (18d) and (31d) have to include the $\delta\hat{q}_{i+1/i}$ terms which are no longer zero.

The latter modifications to the preceding algorithms, which are due to the normalization operation, involve changes which are related to the state. We now ask ourselves whether additional modifications which are related to the covariance matrix are also necessary. To resolve this quandary, let us examine what happens to the estimation error as a result of the normalization operation, because the covariance matrix is the covariance of this error vector. In fact, it will be shown that the estimation error is not affected by the normalization operation. To show this let us first consider the case without normalization. Define the estimation error, after the update of $\delta\hat{q}$, as follows:

$$\epsilon_{i+1/i+1} \triangleq \delta q_{i+1} - \delta\hat{q}_{i+1/i+1} \tag{41}$$

but

$$\delta q_{i+1} = q_{i+1} - \hat{q}_{i+1/i} \tag{42}$$

therefore

$$\epsilon_{i+1/i+1} = q_{i+1} - \hat{q}_{i+1/i} - \delta\hat{q}_{i+1/i+1}. \tag{43}$$

Denote the estimation error, after the reset operation of either (18e) or (31e) is carried out, by $\epsilon'_{i+1/i+1}$ and the corresponding estimate of δq by $\delta\hat{q}'_{i+1/i+1}$. Since the reset is a full one, $\delta\hat{q}'_{i+1/i+1}$ is zero. Then from the definition of the estimation error

$$\epsilon'_{i+1/i+1} \triangleq \delta q'_{i+1} - \delta\hat{q}'_{i+1/i+1} = \delta q'_{i+1}. \tag{44}$$

Now $\delta q'_{i+1}$, the quaternion error after reset, is given by

$$\delta q'_{i+1} = q_{i+1} - \hat{q}_{i+1/i+1} \tag{45}$$

while from either (18e), for the static case, or (31e), for the dynamic case,

$$\hat{q}_{i+1/i+1} = \hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1}. \tag{46}$$

Therefore (45) becomes

$$\delta q'_{i+1} = q_{i+1} - (\hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1}). \tag{47}$$

When (47) is substituted into (44) the following is obtained

$$\epsilon'_{i+1/i+1} = q_{i+1} - \hat{q}_{i+1/i} - \delta\hat{q}_{i+1/i+1}. \tag{48}$$

A comparison between (43) and (48) reveals that the reset operation in the estimation algorithms which does not employ normalization does not affect the estimation error and thus does not affect the covariance matrix either.

In the normalized quaternion estimation algorithm, the expression for the estimation error before normalization is identical to the one obtained for the preceding algorithms. That is, $\epsilon_{i+1/i+1}$ is given, as before, by (43). The estimation error after normalization $\in^*_{i+1/i+1}$ is by definition

$$\epsilon^*_{i+1/i+1} \triangleq \delta q^*_{i+1} - \delta\hat{q}^*_{i+1/i+1} \tag{49}$$

where

$$\delta q^*_{i+1} = q_{i+1} - \hat{q}^*_{i+1/i+1}. \tag{50}$$

Let us denote the matrix $\hat{q}_{i+1/i}\, \hat{q}^{\mathrm{T}}_{i+1/i}$ by C; then following (39) and (40) it is clear that

$$\hat{q}^*_{i+1/i+1} = \hat{q}_{i+1/i} + \delta\hat{q}_{i+1/i+1} - C\, \delta\hat{q}_{i+1/i+1} \tag{51}$$

and

$$\delta\hat{q}^*_{i+1/i+1} = C\, \delta\hat{q}_{i+1/i+1}. \tag{52}$$

Using (50)–(52), (49) becomes

$$\boldsymbol{\epsilon}^*_{i+1/i+1} = \boldsymbol{q}_{i+1} - \hat{\boldsymbol{q}}_{i+1/i} + \delta\hat{\boldsymbol{q}}_{i+1/i+1} \quad (53)$$

which is identical to $\boldsymbol{\epsilon}_{i+1/i+1}$ and $\boldsymbol{\epsilon}'_{i+1/i+1}$ given in (43) and (48), respectively. As mentioned before, the final conclusion is that neither the reset of (18e) or (31e), nor the normalization of (39), changes the estimation error. Therefore, the covariance matrix does not change due to these operations. Consequently, the normalization operation which was introduced in the algorithm for estimating the quaternion does not affect the propagation of the covariance matrix.

The invariance of the estimation error across the normalization operation can be explained by the fact that the sum of the estimated quaternion $\hat{\boldsymbol{q}}$ and the estimated quaternion error $\delta\hat{\boldsymbol{q}}$ stays constant across normalization. This is because the part of $\delta\hat{\boldsymbol{q}}$, which (in order to maintain the normality of $\hat{\boldsymbol{q}}$) is not added to $\hat{\boldsymbol{q}}$, does remain in $\delta\hat{\boldsymbol{q}}$. This feature is accorded to the quaternion estimation since the filter estimates $\delta\boldsymbol{q}$ rather than $\boldsymbol{q}$ directly, which stems from the nonlinear relationship between the quaternion and the DCM. In the DCM estimation algorithm of [6] the situation is different, since there the estimation problem is a linear one and any operation on the estimated DCM, $\hat{D}$, affects the estimation error directly. Therefore, the situation is reversed in the sense that the orthogonalization of the estimated DCM affects the covariance matrix and not the estimated state vector.

The normalized quaternion estimation algorithm for the dynamic case is summarized as follows.

Initial Conditions

$\boldsymbol{q}_0^{\mathrm{T}} = [1,0,0,0]$, see remarks in Section V (54a)

$P_1 = I \cdot a$, I is a 4-dimensional identity matrix and a is a large scalar. (54b)

Between Measurements:

$$\hat{\boldsymbol{q}}_{i+1/i} = \phi_i\, \hat{\boldsymbol{q}}^*_{i/i} \quad (55a)$$

$$P_{i+1/i} = \phi_i\, P_{i/i}\, \phi_i^{\mathrm{T}} + B_i\, Q_i\, B_i^{\mathrm{T}} \quad (55b)$$

$$\delta\hat{\boldsymbol{q}}_{i+1/i} = \phi_i\, \hat{\boldsymbol{q}}_{i/i-1}\, \hat{\boldsymbol{q}}^{\mathrm{T}}_{i/i-1}\, \delta\hat{\boldsymbol{q}}_{i/i}. \quad (55c)$$

Across Measurements:

state update

$$K_{i+1} = P_{i+1/i}\, H^{\mathrm{T}}_{i+1/i} \times (H_{i+1/i}\, P_{i+1/i}\, H^{\mathrm{T}}_{i+1/i} + R_{i+1/i})^{-1} \quad (55d)$$

$$\delta\hat{\boldsymbol{q}}_{i+1/i+1} = \delta\hat{\boldsymbol{q}}_{i+1/i} + K_{i+1}\,(\boldsymbol{e}_{i+1} - H_{i+1/i}\,\delta\hat{\boldsymbol{q}}_{i+1/i}). \quad (55e)$$

estimated quaternion reset and normalization

$$\hat{\boldsymbol{q}}_{i+1/i+1} = \hat{\boldsymbol{q}}_{i+1/i} + \delta\hat{\boldsymbol{q}}_{i+1/i+1} \quad (55f)$$

$$\hat{\boldsymbol{q}}^*_{i+1/i+1} = \hat{\boldsymbol{q}}_{i+1/i+1} / \|\hat{\boldsymbol{q}}_{i+1/i+1}\|. \quad (55g)$$

covariance update

$$P_{i+1/i+1} = (I - K_{i+1}\, H^*_{i+1/i+1})\, P_{i+1/i} \times (I - K_{i+1}\, H^*_{i+1/i+1})^{\mathrm{T}} + K_{i+1}\, R^*_{i+1/i+1}\, K^{\mathrm{T}}_{i+1}. \quad (55h)$$

The asterisk on H and R denote that they are recomputed using $\hat{\boldsymbol{q}}^*_{i+1/i+1}$. The algorithm for the static case is of course a special case of the latter, and is obtained by replacing ϕ_i with the identity matrix.

V. MONTE-CARLO SIMULATION

To examine the performance of the normalized quaternion estimation algorithm versus the performance of the ordinary quaternion estimation algorithm in terms of convergence and orthogonality, we define two indices of performance. The convergence index of the normalized algorithm is defined as

$$J^*_{i+1} = \mathrm{tr}[(\hat{D}^*_{i+1/i+1} - D_{i+1})^{\mathrm{T}}\,(\hat{D}^*_{i+1/i+1} - D_{i+1})] \quad (56)$$

where $\hat{D}^*_{i+1/i+1}$ and D_{i+1} are the DCMs which correspond to $\hat{\boldsymbol{q}}^*_{i+1/i+1}$ and $\boldsymbol{q}_{i+1}$, respectively. The convergence index J of the ordinary algorithm is defined in a similar way except that $\hat{D}_{i+1/i+1}$ replaces $\hat{D}^*_{i+1/i+1}$, where $\hat{D}_{i+1/i+1}$ is the DCM which corresponds to $\hat{\boldsymbol{q}}_{i+1/i+1}$.

The orthogonality index is defined for the normalized case as

$$F^*_{i+1} = \mathrm{tr}[(\hat{D}^{*\mathrm{T}}_{i+1/i+1}\,\hat{D}^*_{i+1/i+1} - I)^{\mathrm{T}} \times (\hat{D}^{*\mathrm{T}}_{i+1/i+1}\,\hat{D}^*_{i+1/i+1} - I)]. \quad (57)$$

For the ordinary case we use the same definition, but without the asterisks, to obtain F_{i+1}. Note that the convergence index measures the closeness of the estimated DCM to the actual one; it is always positive and becomes zero for a perfect convergence. Similarly, the orthogonality index measures the closeness of the estimated DCM to orthogonality; it is always positive and becomes zero only when the estimated DCM is orthogonal. (Note that a normal quaternion automatically yields an orthogonal matrix.)

Since the quaternion propagation $\boldsymbol{u}_i$, $\boldsymbol{v}_i$ and thus the quaternion estimate are random processes, it is not enough to show results for only one time simulation run. Therefore 100 Monte-Carlo simulation runs were carried out for each of the two estimation algorithms. The expected values of their indices were computed as follows

$$\bar{J}^*_i \triangleq E\{J^*_i\} \sim \frac{1}{100}\sum_{k=1}^{100} J^*_i(k) \quad (58a)$$

$$\bar{J}_i \triangleq E\{J_i\} \sim \frac{1}{100}\sum_{k=1}^{100} J_i(k) \quad (58b)$$

$$\bar{F}^*_i \triangleq E\{F^*_i\} \sim \frac{1}{100}\sum_{k=1}^{100} F^*_i(k) \quad (59a)$$

$$\bar{F}_i \triangleq E\{F_i\} \sim \frac{1}{100}\sum_{k=1}^{100} F_i(k). \tag{59b}$$

In each run the initial DCM, D_0, was

$$D_0 = \begin{bmatrix} 0.33696 & -0.88924 & 0.30937 \\ 0.18352 & -0.26025 & -0.94794 \\ 0.92346 & 0.37620 & 0.07550 \end{bmatrix} \tag{60}$$

whereas the initial quaternion estimate was $\boldsymbol{q}_0^{\mathrm{T}} = [1,0,0,0]$, which meant that the estimate of the initial DCM was the identity matrix. The nominal angular rate was a constant vector whose components in body axes were 0.628 rad/s along each axis. The gyro measurement noises which constitute $\delta\boldsymbol{\omega}$ were three, zero-mean white-noise components, whose spectral density was 0.01 o/h$^{1/2}$. A measured pair, $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$, was obtained and processed every 0.1 s. The measurement noise of $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ was zero mean and white. Its standard deviation corresponded to a random angular error of 100 arcsec.

Plots of the convergence indices are shown in Fig. 1. It is clearly seen that $\bar{J}_i$, which is the time average of the convergence index of the untuned ordinary quaternion estimation algorithm, diverges after reaching a minimum point. Tuning of this filter solved the problem as can be seen from the plot of $\bar{J}'_i$, the corresponding averaged convergence index. Tuning was achieved by replacing $B_iQ_iB_i^{\mathrm{T}}$, in (31b) by a matrix whose value was adjusted to achieve the lowest steady-state value of $\bar{J}_i$. The initial value of that matrix was chosen as the value which preserved the value of $P_{i+1/i}$ at the level which $P_{i+1/i}$ had during the time segment in which the untuned filter exhibited fast convergence. The plot of $\bar{J}_i$ demonstrates the better accuracy achieved when the normalized estimation algorithm was used. The corresponding orthogonality indices shown in Fig. 2 exhibit a similar behavior. Since the normalized quaternion estimation algorithm yields an orthogonal DCM after each normalization operation, $\bar{F}_i^*$ is practically zero and is not plotted in Fig. 2. The effect of the normalization of the estimated quaternion is more pronounced than that of the orthogonalization of the estimated DCM, which is

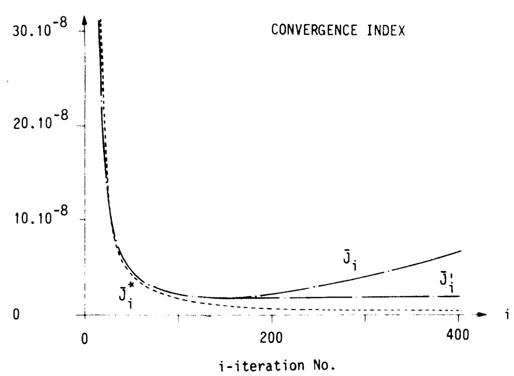


Fig. 1. Ensemble average of convergence index of quaternion estimation algorithm with normalization ($\bar{J}_i^*$), without normalization ($\bar{J}_i$), and without normalization but with tuning ($\bar{J}'_i$).

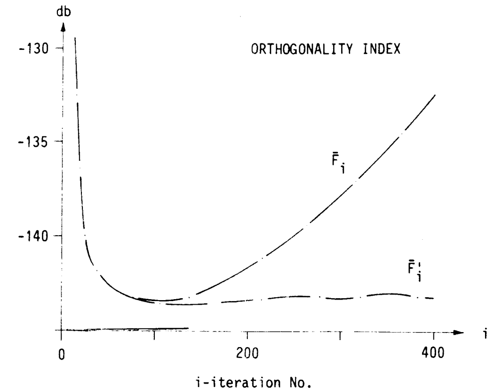


Fig. 2. Ensemble average of orthogonality index of quaternion estimation algorithm with normalization ($\bar{F}_i$) and without normalization but with tuning ($\bar{F}'_i$).

performed in [6]. This is because the quaternion normalization algorithm is exact while the DCM orthogonalization algorithm is just one cycle of an iterative process.

The algorithms were tested for several true initial attitudes and although the initial DCM estimate was always the identity matrix, the algorithms invariably behaved in a manner similar to the one presented in Fig. 1.

VI. CONCLUSION

Two quaternion estimation algorithms were presented which process a sequence of pairs of vector measurements. Since the functional relationship between the measured vector pairs and the estimated quaternion is nonlinear, it was necessary to develop EKF-like algorithms to estimate the difference between the true quaternion and its estimate. The estimate of the difference was added to the whole quaternion estimate each time the former was updated after the acquisition of new measurements. The two algorithms deviate from the classical EKF mainly in that their covariance update uses the current quaternion estimate. In addition, the normalized quaternion estimation version includes a normalization stage which is a peculiar operation. The normalized quaternion estimation algorithm makes use of the fact that the quaternion of rotation is normal to upgrade the first algorithm presented in this work. The result is an algorithm which converges to a more accurate completely normal quaternion. While the latter is obtained through a straightforward design, the nonnormalized version requires filter tuning.

Monte-Carlo simulation runs were carried out to test the algorithms. True body rotations were simulated, measurements were generated using random number generators, and the estimation algorithms were applied to the simulated measurements in order to estimate the time varying quaternion of rotation. The difference between the true and the estimated DCMs was used to evaluate convergence and orthogonality indices. One hundred simulation runs were made and their indices were averaged at each update point to obtain an estimate of the expected value of the indices. These averages were examined in order to evaluate the algorithms. It was found that both algorithms always converged independent of the initial orientation, although in all cases the initial estimate was $\boldsymbol{q}_0^{\mathrm{T}} = [1,0,0,0]$. The normalized quaternion estimation version, which yields a normal estimate, exhibited better convergence properties.

As a result of the Monte-Carlo simulation runs it is recommended that the normalized version of the algorithm be used. Finally, it is recommended that further work be performed to compare the latter algorithm with the one presented in [10].

REFERENCES

[1] Lerner, G.M. (1978)
Three-Axis attitude determination.
Spacecraft Attitude Determination and Control, (J.R. Wertz, ed.) Dordrecht, The Netherlands: D. Reidel Publishing Co., 1978, pp. 420–428.

[2] Shuster, M.D., and Oh, S.D. (1981)
Three axis attitude determination from vector observations.
AIAA Journal of Guidance and Control, 4 (Jan.–Feb. 1981), 70–77.

[3] Wahba, G., Farrel, J.L., and Stuelpnagel, J.C., Wessner, B.H., Velgam, J.R., Brock, I.E., and Desjardins, R. (1966)
Problem 65-1 (solution).
SIAM Review, 8 (1966), 384–386.

[4] Brock, J.E. (1968)
Optimal matrices describing linear systems.
AIAA Journal, 6 (July 1968), 1292–1296.

[5] Carta, D.G., and Lackowski, D.H. (1972)
Estimation of orthogonal transformations in strapdown inertial systems.
IEEE Transactions on Automatic Control, AC-17 (Feb. 1972), 97–100.

[6] Bar-Itzhack, I.Y., and Reiner, J. (1982)
Recursive attitude determination from vector observations: DCM identification.
Technical Report TAE 501, Department of Aeronautical Engineering, Technion Israel Institute of Technology, Oct. 1982.

[7] Ickes, B.P. (1970)
A new method for performing digital control system attitude computations using quaternions.
AIAA Journal, 8 (Jan. 1970), 13–17.

[8] Shuster, M.D. (1978)
Approximate algorithms for fast optimal attitude computation.
In *Proceedings, AIAA Guidance and Control Conference* (Palo Alto, Calif., Aug. 7–9, 1978), Paper 78-1249.

[9] Lefferts, E.J., Markley, F.L., and Shuster, M.D. (1982)
Kalman filtering for spacecraft attitude estimation.
AIAA Journal on Guidance, Control, and Dynamics, 5 (Sept.–Oct. 1982), 417–429.

[10] Gai, E.G., Lemos, L.K., Harrison, J.V., and Daly, K.C.
Attitude and rate estimator using only stellar data.
To be submitted to *AIAA Journal Guidance, Control and Dynamics.*

[11] Wilcox, J.C. (1967)
A new algorithm for strapped-down inertial navigator.
IEEE Transactions on Aerospace and Electronic Systems, AES-3 (Sept. 1967), 796–802.

[12] Nash, R.A., Jr., D'Appolito, J.A., and Roy, K.J. (1972)
Error analysis of hybrid aircraft inertial navigation systems.
In *Proceedings, AIAA Guidance and Control Conference* (Stanford, Calif., Aug. 14–16, 1972), Paper 72-848.

[13] Gelb, A. (Ed.) (1974)
Applied Optimal Estimation.
Cambridge, MA: MIT Press, 1974.

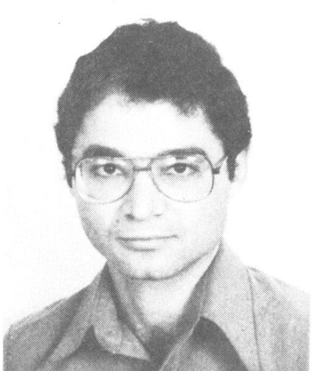
[14] Maybeck, P.S. (1979)
Stochastic Models, Estimation, and Control.
New York: Academic Press, 1979

[15] Bar-Itzhack, I.Y. (1971)
Optimum normalization of a computed quarternion of rotation.
IEEE Transactions on Aerospace and Electronic Systems. AES-7, Mar. 1971, 401–402.

Itzhack Y. Bar-Itzhack (M'73) was born in Israel on August 19, 1937. He received the B.Sc. and M.Sc. degrees in electrical engineering from the Technion—Israel Institute of Technology, Haifa, in 1961 and 1964, respectively. In 1964 he came to the United States and received the Ph.D. degree in electrical engineering from the University of Pennsylvania, Philadelphia, in 1968.

From 1961 to 1964 he was a Teaching and Research Assistant at the Technion—Israel Institute of Technology in the field of automatic control. In 1965 he became a Research Assistant at the Moore School of Electrical Engineering, University of Pennsylvania, where he did work on strapdown inertial navigation systems. From 1968 to 1971 he was a member of the Technical Staff at Bellcomm, Inc., Washington, D.C., where he worked on the Saturn V longitudinal stability problem and on the Lunar Roving Vehicle navigation system. In 1971 he joined the Aeronautical Engineering Department of the Technion—Israel Institute of Technology where he is an Associate Professor. During the 1977–78 academic year he spent his sabbatical with The Analytic Sciences Corporation (TASC), Reading, Mass., where he worked in research and development of multisensor integrated navigation. His research interests include inertial navigation, estimation, and guidance.

Dr. Bar-Itzhack is a member of Sigma Xi.



Yaakov Oshman was born in Israel on November 4, 1953. He received his B.Sc. degree in aeronautical engineering from the Technion—Israel Institute of Technology, Haifa, in 1975. After his military service as an Engineer with the Israeli Air Force, he started his graduate work in 1981 at the Aeronautical Engineering Department of the Technion, where he is currently pursuing a D.Sc. program. His research interests include numerical solutions of the matrix Riccati equation, square root and reduced order filtering.